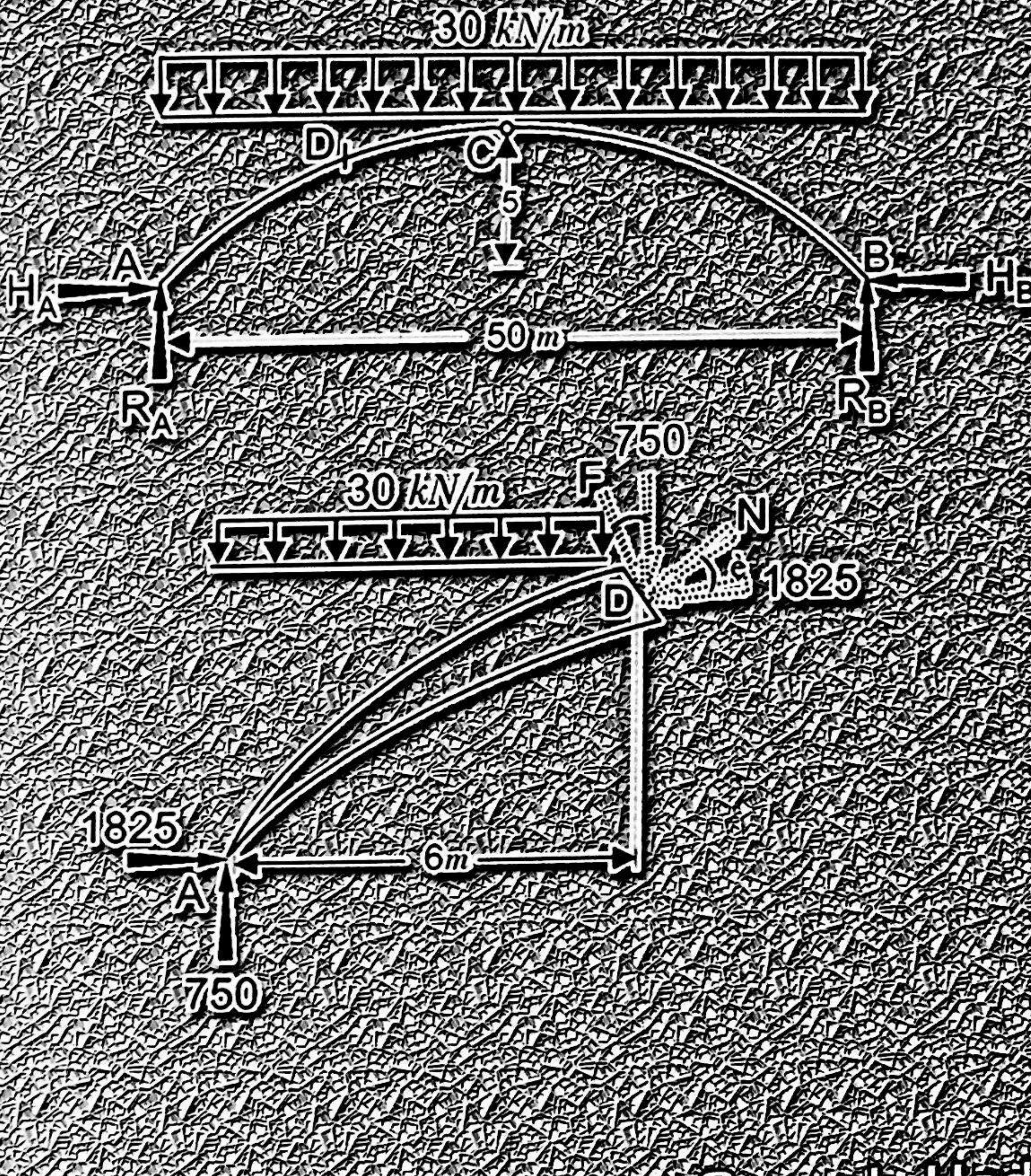
text book on

THEORY OF STRUCTURES



Suresh Hada

CONTENTS

SYMBOL AND NOTATIONS

A	Area
BM	Bending moment
C_1, C_2	Integration constants
d	Difference between height of supports
da	differential area
dv	Differential volume
dx	Small length in x direction
dy	Small length in y direction
E	Modulus of Elasticity
Eq.	Equation
F	Shear force, Radial shear
F_{AB}	Force in member AB
F^*	Unit virtual force
ΣF_x	Sum of forces in x direction
ΣF_y	Sum of forces in y direction
G	Modulus of rigidity, Centre of gravity
h	Height, Crown height of an arch
I	Moment of inertia
J	Polar moment of inertia
K	Constant depending on the shape of a section
$\Sigma M = 0$	Sum of moments
M_{abs}	Absolute moment
M^*	Unit virtual moment
M_x	Moment at x
N	Normal thrust
P, W	Point load
R	Radial shear
R_A	Reaction at A
rad.	Radian
R_B	Reaction at B
S	Length of cable
SF	Shear force
t	Thickness
T	Tangent, Torque
T_{max}	Maximum thrust
U	Strain energy, Ultimate point
U_0	Strain energy density
V	Vertical shear force

x	Distance from support to a section considered
y	Distance from support to a load, deflection
y_C	Deflection at C
z	Position of unit load from left support
δ, Δ	Deflection
δ_{st}	Static deflection
δ_d	Dynamic deflection
ε	Strain
γ	Shear strain
θ	Slope
θ_A	Slope at A
θ_B	Slope at B
ρ	Density of material
σ	Stress
τ	Shear stress
w	Uniformly distributed load
ψ	Angle made by tangent with x axis
ℓ	Span
	Hinged support
	Roller support
	Fixed support
	Reaction forces at support, applied load
	Reaction moment at support, applied moment
	Internal forces developed, rolling load
	Internal moment developed

1

INTRODUCTION TO ANALYSIS OF STRUCTURES

1.1 STRUCTURE AND ITS DEVELOPMENT

A structure is a body made up of an assemblage of members like bars, plates, walls etc. connected together such that they can resist the applied load without appreciable deformation. Structures are created to serve some specific functions like human habitation, bridges, warehouses etc. The caves and hollow tree are probably the first structures when early humans lived in. But often these natural shelters were not properly situated with respect to the source of food and water. So, these early people started to build their first structures near the places where these sources of food and water were nearby.

In ancient and medieval periods, due to unavailability of proper mathematical and scientific knowledge, the method of construction of structures existed just as an art. Later on with the development of fundamentals of mechanics, it took the form of structural engineering.

Structural Engineering or Theory of Structures is one of the most important subjects of Civil Engineering. It deals with the principles and methods by which the internal forces (Axial force, Shear force, Bending moment etc.) and their effect (Deflection) developed due to the application of external force/s in any constituent members of a structure may be calculated. The determination of these internal forces is necessary for designing Structures.

1.2 TYPES OF STRUCTURES

Broadly, structures can be classified according to the nature of their internal forces, material being used and the method of analysis employed. As per the nature of internal forces, structures can be classified as uniform stress and varying stress form. In the former, every element of the structure is subjected to a uniform stress. Thus if the stress at a point of the structure becomes critical, the stress in all parts of the structure will also be critical. In the latter, non-uniform stress distribution occurs. As a result even if the stresses at certain points of the structure approach the critical value, the stress in the other parts of the structure may still be much below the critical value. It is obvious that the uniform stress form makes more efficient use of material.

The example of varying stress forms are beams and slabs whereas those of uniform stress form are cables, arches, shells and truss structures.

When a slender member is supported and loaded laterally as shown in Fig.(1.1-a) it is subjected to shear and bending and is called a beam. Under the condition, the shear and bending stresses vary from point to point along the length. If a plate is supported and loaded laterally, in general, it is subjected to shear, bending and twisting and is called a slab. (Fig. 1.2-b). The forces vary along the length and width of the slab. These varying stress structural forms are used in construction of building and bridges. Although full materials are not used in this type of structural forms, they are often adopted due to functional and economical reasons.

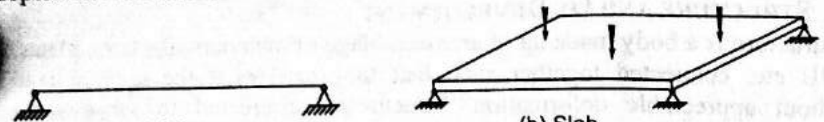


Fig. 1.1

Flexible structural forms can not resist compression, shear and bending. Only tensile reaction forces are developed under the application of load. While a cable is loaded with uniformly distributed load along the span, it takes the shape of a parabola Fig. (1.2). Since the slope of such cable varies and the horizontal component of the tension in the cable remain constant along its length, the tensile force along the cable varies from point to point along its length.

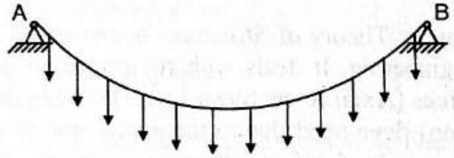


Fig. 1.2

An arch by its form is similar to an inverted cable. It is not a flexible structure as the cable is, and hence posses flexural strength. It essentially resists the external load by developing a uniform stress condition across its sections. A three-hinged arch loaded with uniformly distributed load along its span is an example in which a perfect axial force is developed Fig. (1.3). For any other type of loading, shear and bending as well as thrust exist at a section normal to the axis of the arch.

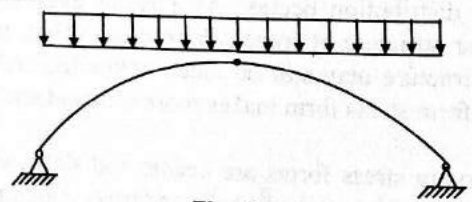


Fig. 1.3

A shell is a three dimensional structure having flexural stiffness and strength. It normally carries load by developing compressive stress rather than tensile stress.

Another example of uniform stress structural form is the planar truss. A truss can be defined as a structural form consisting of members jointed at their extremities to form triangular shape (Fig. 1.4). The joints of a truss are pin jointed and the nature of the force in the members is either tensile or compressive.

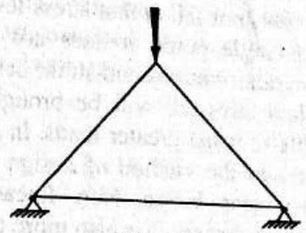


Fig. 1.4

Structures can also be classified according to the materials being used. A number of structures come under this classification eg. Steel, Concrete, Masonry, Wooden and Plastic structures. Likewise structures can be distinguished according to the method of analysis. Structures whose unknown reaction forces can be determined by using three equilibrium equations only, they are called determinate structures otherwise they are called indeterminate structures. Figure (1.5-a) and (b) represents determinate and indeterminate structures respectively.

Fig. 1.5

1.3 PURPOSE OF ANALYSIS

Broadly speaking, the process of design of a structure consists of two stages. The first stage deals with the determination of forces at any section of a structure. The second stage is the selection and design of suitable section to resist the load that comes over the structure in its life period. The first stage is called the analysis and the second stage is called the actual design. Thus structural analysis is the first step, which is necessary for designing a structure.

Analysis of a structure specifically means the determination of stress resultants (Reaction forces, Bending moment, Shear force etc.) at every section. The calculation of these reaction, shear and moment and stresses is

an important step in the design. Engineers use the fundamental principles of structural analysis to determine such forces.

1.4 Linearity and Non Linearity

When stress and strain relationship for a structure or a component of a structure obeys Hook's law, the structure is called linearly elastic structure. The behavior of such a structure is simple for analysis and it is assumed that when a section of a structure attends its yield stress, the structure fails. But actually, the structure does not fail at that stress level. In other words, if yield point is attained at a single point, it does not mean the collapse of the member. Due to plastic deformations and strain hardening of the material, the particles, which are less stressed, will be brought into action so that the structure actually is able to resist greater loads. In modern designs, the above principle is introduced and the method of design by this principle is called plastic design or non- linear design. Non- linear analysis is necessary to proceed to such non- linear design. It is also more complicated than the linear design but is economical as the reserve strength beyond the elastic limit is utilized.

The linear and non-linear stress strain curve for mild steel is shown in Fig. (1.6).

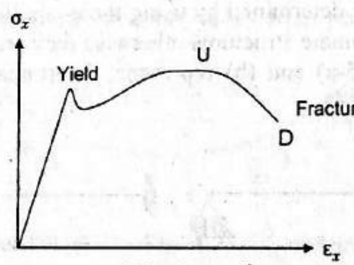


Fig. 1.6

1.5 Method of Structural Analysis

As stated earlier, structural analysis deals with the principles and method by which the axial force, shear and bending moment at any section of the member may be found under the given condition of loading. Because the forces acting on a structural member may usually be assumed to lie in the same plane and are in equilibrium, fundamental structural analysis involves the use of the three equations of equilibrium for a general coplanar force system eg. $\Sigma F_x = 0$, $\Sigma F_x = 0$, and $\Sigma M = 0$.

In analyzing indeterminate structures, the above three equilibrium equations are not sufficient. Two methods of analysis are generally employed in such case namely: Force method and Displacement method.

The force method consists of writing equations that satisfy the compatibility and force displacement requirements for a structure and involve redundant forces as unknowns. Once the redundant forces have been determined, the remaining forces are determined with the help of the three equilibrium equations.

Displacement method on the other hand, consists of writing force displacement relation first and then satisfying equilibrium requirements for the structure. In this case, the unknowns in the equations are displacements. Once the displacements are obtained, the forces are determined from the compatibility and force displacement equations.

2

DEFLECTION OF BEAMS

2.1 INTRODUCTION

Whenever a beam is loaded, it deflects from its initial position. The deflection disappears when the load is removed provided that the elastic limit of the material is not exceeded. Deflection in a structure is also caused by various reasons such as change in temperature, lack of fit of members, creep, settlement of supports etc.

The computation of deflection is necessary for many reasons. A beam designed primarily for strength should also fulfill stiffness criteria. It means that in addition to the strength, the beam should be stiff enough not to deflect more than the given limit under the load. Deflection computation is thus needed for a designer to check whether the beam has exceeded the deflection limit. Moreover, the computation of deflection is also required for analyzing statically indeterminate structures.

2.2 DEFLECTION, SLOPE AND CURVATURE OF A BEAM

When a load is applied on a beam, its neutral axis bends into a curved line from its original position. The vertical ordinate between the curve and the initial neutral axis gives the deflection of the beam at the section. In the Fig. (2.1), y is the deflection.

Slope of a beam at a point is an angle made by a tangent to the referece axis at that point. θ_A and θ_B represent the slopes at A and B respectively. The distance OC is the radius of curvature R and the inverse of it is called the curvature. The deflection, slope and curvature are shown in the figures given below.

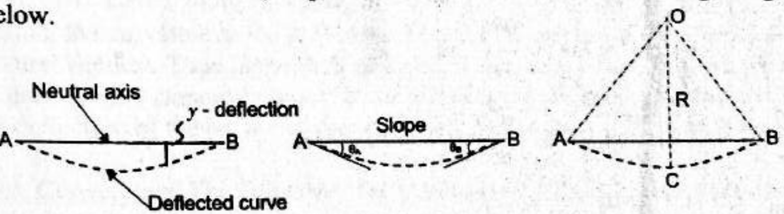


Fig. 2.1

2.3 RELATION BETWEEN SLOPE, DEFLECTION AND CURVATURE

Refering Fig. (2.2), let mn be a segment of a bent beam of length ds which subtends an angle $d\psi$ at the centre C. Let (x, y) and $(x + dx,\ y + dy)$ are the coordinates of m and n and T_1 and T_2 are the tangents at m and n which subtends angles ψ and $\psi + d\psi$ respectively at x axis.

From the geometry, we have,

$\angle mcn = d\psi$ and $ds = R.d\psi$ ....(2.1)

Now,

$$ds = \sqrt{dx^2 + dy^2}$$

$$= dx\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}$$

and $\tan\psi = \dfrac{dy}{dx}$

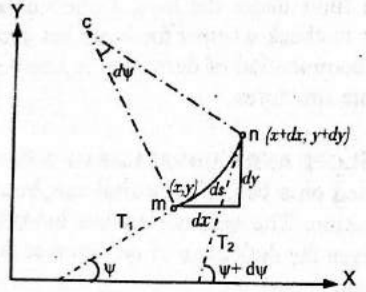


Fig. 2.2

Differentiating,

$$\sec^2\psi.d\psi = \frac{d^2y}{dx^2}.dx$$

or, $(1+\tan^2\psi).d\psi = \dfrac{d^2y}{dx^2}.dx$

or, $\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}.d\psi = \dfrac{d^2y}{dx^2}.dx$

$\therefore\quad d\psi = \dfrac{\dfrac{d^2y}{dx^2}.dx}{1+\left(\dfrac{dy}{dx}\right)^2}$

Substitute the value of $d\psi$ and ds in Eq. (2.1) we get,

$$dx\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} = \frac{R\dfrac{d^2y}{dx^2}.dx}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^1}$$

or, $\dfrac{1}{R} = \dfrac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}$

As $\dfrac{dy}{dx}$ is small, $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2} \approx 1$

$\therefore\quad \dfrac{1}{R} = \dfrac{d^2y}{dx^2}$

Also, we have,

$$\frac{M}{I} = \frac{E}{R} \text{ or } \frac{1}{R} = \frac{M}{EI}$$

$\therefore\quad \dfrac{M}{EI} = \dfrac{d^2y}{dx^2}$

or, $EI.\dfrac{d^2y}{dx^2} = M$ ....(2.2)

Integrating,

$EI.\dfrac{dy}{dx} = \int M$ ....(2.3)

$EI.y = \iint M$ ....(2.4)

Eq.(2.2) is called moment curvature relationship. If M is the moment at a section, the curvature at the section is d^2y/dx^2. The quantity EI is referred to as flexural rigidity. Thus integration of Eq. (2.2) leads to {Eq. (2.3)} which can be used to find slope of a beam. If the slope equation is integrated again, we get deflection of the beam as given by Eq. (2.4).

Sign Convention: The following sign convention will be used to solve the problems related to deflection.

- x is positive when measured towards right.
- y is positive when measured upwards.
- M (bendng moment) is positive when sagging.
- θ (slope) is positive when the rotation is anticlockwise.

2.4 Slope and Deflection Computations

There are number of methods to compute deflections of a beam. However, following important methods will be discussed here.

- Double integration method
- Macaulay's method
- Moment area method
- Conjugate beam method
- Virtual work method (Unit load method)

2.5 Double Integration Method

In this method the moment curvature relationship Eq. (2.2) is integrated for finding slope and deflection at a point of any beam. In this section, the expressions for slope and deflections for simple beams are derived first and the related numericals are presented later.

a) Cantilever with a point load at the free end

Refering Fig. (2.3), consider a section at a distance x from the free end of the cantilever beam.

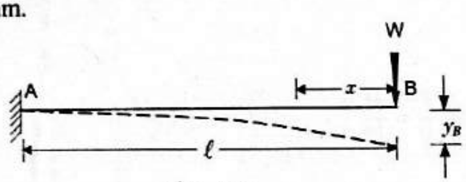


Fig. 2.3

Bending moment at the section x is

$$M_x = -Wx \qquad \text{(–ve sign due to hogging moment)}$$

or, $$EI\frac{d^2y}{dx^2} = -Wx$$

Integrating,

$$EI\frac{dy}{dx} = -\frac{Wx^2}{2} + C_1 \qquad \text{.................................(2.5)}$$

(C_1 is integration constant)

Applying the boundary condition,

$$\frac{dy}{dx} = 0 \text{ at } x = \ell,$$

we get,

$$0 = -\frac{W\ell^2}{2} + C_1 \text{ or } C_1 = \frac{W\ell^2}{2}$$

Substituting the value of C_1 in Eq. (2.5),

$$EI\frac{dy}{dx} = -\frac{Wx^2}{2} + \frac{W\ell^2}{2} \qquad \text{.................................(2.6)}$$

This is the equation to find slope of the beam at any point. The maximum slope in the beam occurs at the free end B where $x = 0$. If θ_B is the slope at B, then,

$$EI\frac{dy}{dx} = 0 + \frac{W\ell^2}{2}$$

$$EI\,\theta_B = \frac{W\ell^2}{2}$$

$$\therefore \quad \theta_B = \theta_{max} = \frac{W\ell^2}{2EI} \text{ radians} \qquad \text{.................................(2.7)}$$

Integrating Eq. (2.6) once again,

$$EI.y = -\frac{Wx^3}{6} + \frac{W\ell^2 x}{2} + C_2 \qquad \text{.................................(2.8)}$$

(C_2 is integration constant)

Applying boundary condition, $y = 0$ at $x = \ell$, we get,

$$0 = -\frac{W\ell^3}{6} + \frac{W\ell^2.\ell}{2} + C_2 \qquad \text{or, } C_2 = -\frac{W\ell^3}{3}$$

Substituting the value of C_2 in Eq. (2.8),

$$EI.y = -\frac{Wx^3}{6} + \frac{W\ell^2 x}{2} - \frac{W\ell^3}{3}$$

This is the equation to find deflection at any point of the beam. Maximum deflection in the beam occurs at the free end B where $x = 0$. If y_B is the deflection at B, then

$$EI.y_B = -\frac{W\ell^3}{3}$$

or, $$y_B = -\frac{W\ell^3}{3EI} \qquad \text{.................................(2.9)}$$

(– ve sign shows the deflections in the downward direction)

b) Cantilever with a point load not at the free end

Refering Fig. (2.4), consider a section at a distance x from the free end B.

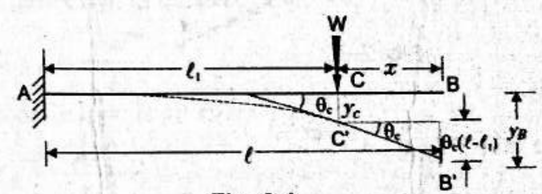


Fig. 2.4

Here the portion AC of the cantilever will bend into AC' while the portion CB will remain straight. The problem will be treated, as a cantilever of length AC with an end load W at C and its slope and deflection will be same as of the previous problem. Once y_c is determined, y_b will be found out by considering geometry. As from the previous section,

$$\theta_C = \frac{W\ell_1^2}{2EI} \quad \text{(From Eq. 2.7)}$$

Since the portion CB of the cantilever is straight,

$$\theta_B = \theta_C = \frac{W\ell_1^2}{2EI} \quad \text{and}$$

$$y_c = \frac{W\ell_1^3}{3EI}$$

From the geometry of the diagram (Fig 2.4)

$$y_B = y_C + \theta_C(\ell - \ell_1) = \frac{W\ell_1^3}{3EI} + \frac{W\ell_1^2}{2EI}(\ell - \ell_1) \quad \text{.............. (2.10)}$$

c) Cantilever with several point loads

When the numbers of forces are acting slope and deflections at the sections of the beam are calculated as a sum of slope and deflections due to each of the forces acting. These are then superimposed as shown in the Fig. (2.5).

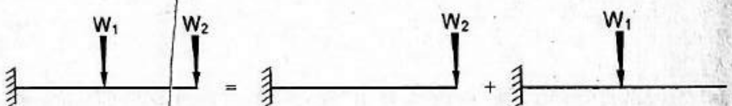


Fig. 2.5

This problem can now be solved by combining the equations given above.

d) Cantilever with a uniformly distributed load

Refering Fig. (2.6), consider a section at x- from the free end B.

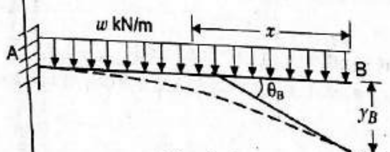


Fig. 2.6

Moment at distance x from B is $M_x = -\dfrac{wx^2}{2}$

or, $$EI\frac{d^2y}{dx^2} = -\frac{wx^2}{2}$$

Integrating,

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + C_1 \quad \text{...(2.11)}$$

(C_1 is the constant of integration)

Applying the boundary condition, $\dfrac{dy}{dx} = 0$ at $x = \ell$

$$0 = \frac{-w\ell^3}{6} + C_1$$

$$\therefore \quad C_1 = \frac{w\ell^3}{6}$$

Substituting this in Eq. (2.11)

$$EI\frac{dy}{dx} = \frac{-wx^3}{6} + \frac{w\ell^3}{6} \quad \text{.. (2.12)}$$

This is the equation used to find slope at any section of the beam. The maximum slope in the beam occurs at the free end B when $x = 0$. If θ_B is the slope at B, then

$$EI.\theta_B = \frac{w\ell^3}{6} \text{ radians} \quad \text{.. (2.13)}$$

Integrating Eq. (2.12), agian

$$EI.y = \frac{-wx^4}{24} + \frac{w\ell^3 x}{6} + C_2 \quad \text{.. (2.14)}$$

(C_2 is integration constant)

Applying boundary condition, $y = 0$ at $x = \ell$

$$0 = \frac{-w\ell^4}{24} + \frac{w\ell^4}{6} + C_2, \quad \text{or} \quad C_2 = \frac{-w\ell^4}{8}$$

Substituting the value of C_2 in Eq. (2.14)

$$EI.y = \frac{-wx^4}{24} + \frac{w\ell^3 x}{6} - \frac{w\ell^4}{8} \quad \text{.................................. (2.15)}$$

This is the equation used to find deflection at any section of the beam. The maximum deflection in the beam occurs at free end where $x = 0$

$$\therefore \quad EI.y_B = \frac{-w\ell^4}{8}$$

Maximum deflection is given by

$$y_B = -\frac{w\ell^4}{8EI} \quad \text{...(2.16)}$$

(– ve sign indicate that the deflection is downward)

e) Cantilever with a moment applied at the free end

Refering Fig. (2.7), consider a section at a distance x from the free end B.

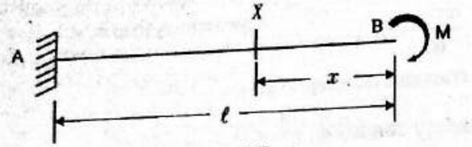


Fig. 2.7

We have,

$$M_x = -M$$

or, $$EI\frac{d^2y}{dx^2} = -M$$

Integrating,

$$EI.\frac{dy}{dx} = -Mx + C_1 \quad \text{.......... (2.17)}$$

(C_1 is constant of integration)

Applying boundary condition, $\frac{dy}{dx} = 0$ at $x = \ell$

or, $0 = -M\ell + C_1$ or, $C_1 = M\ell$

Substituting the value of C_1 in Eq. (2.17)

$$EI\frac{dy}{dx} = -Mx + M\ell \quad \text{.......... (2.18)}$$

This is the equation to find slope of the beam at any point. The maximum slope in the beam occurs at its free end B where $x = 0$. If θ_B is the slope at B then Eq. (2.18) gives,

$$EI.\theta_B = M\ell$$

or, $$\theta_B = \theta_{max} = \frac{M\ell}{EI} \quad \text{.......... (2.19)}$$

Integrating Eq. (2.18) again,

$$EI.y = \frac{-Mx^2}{2} + M\ell x + C_2 \quad \text{.......... (2.20)}$$

Applying boundary condition, $y = 0$, at $x = \ell$

$$0 = \frac{-M\ell^2}{2} + M\ell^2 + C_2$$

or, $$C_2 = -\frac{M\ell^2}{2}$$

Substituting the value of C_2 in Eq. (2.20)

$$EI.y = \frac{-Mx^2}{2} + M\ell x - \frac{M\ell^2}{2} \quad \text{.......... (2.21)}$$

This is the equation to find deflection of the beam at any point of the beam. Maximum deflection in the beam occurs at the free end B where $x = 0$. If y_B is the deflection at B, then,

$$EI.y_B = \frac{-M\ell^2}{2}$$

$$y_B = \frac{-M\ell^2}{2EI} \quad \text{.......... (2.22)}$$

f) Cantilever with a gradually varying load

Refering Fig. (2.5.6), consider a section at x distance from the free end,

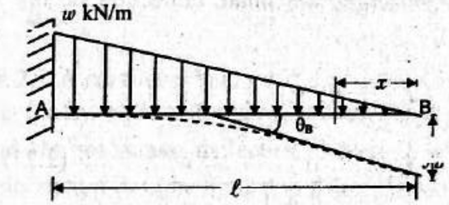


Fig. 2.8

$$M_x = \frac{-1}{2} \times x \times \frac{wx}{\ell} \cdot \frac{x}{3} = \frac{-wx^3}{6\ell}$$

or, $$EI.\frac{d^2y}{dx^2} = \frac{-wx^3}{6\ell}$$

Integrating,

$$EI.\frac{dy}{dx} = \frac{-wx^4}{24\ell} + C_1 \quad \text{.......... (2.23)}$$

(C_1 is integration constant)

Applying boundary condition, $\frac{dy}{dx} = 0$ at $x = \ell$

or, $$0 = \frac{-w\ell^4}{24\ell} + C_1$$

$C_1 = \frac{w\ell^3}{24}$, substituting this in Eq. (2.23),

$$EI.\frac{dy}{dx} = \frac{-wx^4}{24\ell} + \frac{w\ell^3}{24} \quad \text{.......... (2.24)}$$

This is the equation for maximum slope that occurs at $x = 0$ substituting this,

or, $$EI.\theta_B = \frac{w\ell^3}{24}$$

$$\theta_B = \frac{w\ell^3}{24EI} \qquad \text{.. (2.25)}$$

Integrating the Eq. (2.24) once again,

$$EI.y = -\frac{wx^5}{120\ell} + \frac{w\ell^3 x}{24} + C_2 \qquad (C_2 \text{ is integration constant})$$

Applying boundary condition, $y = 0$ at $x = \ell$

$$0 = -\frac{w\ell^4}{120} + \frac{w\ell^4}{24} + C_2, \quad \text{or,} \quad C_2 = \frac{-w\ell^4}{30}$$

$$\therefore \quad EI.y = \frac{-wx^5}{120\ell} + \frac{w\ell^3 x}{24} - \frac{w\ell^4}{30} \qquad \text{.......................... (2.26)}$$

This is the equation for deflection. Maximum deflection occurs at $x = 0$, substituting this value.

or, $$EI.y_B = \frac{-w\ell^4}{30}$$

or, $$y_B = -\frac{w\ell^4}{30EI} \qquad \text{.. (2.27)}$$

Example # 2.1 Find the maximum slope and deflection of the cantilever beam loaded as shown in Fig. (2.9) where $AB = CB = 1\ m$. Use the integration method. Take $E = 2\times10^5\ N/mm^2$, $I = 1.5\times10^8\ mm^4$.

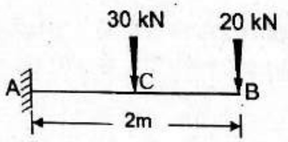


Fig. 2.9

Sol^n. Maximum slope and deflection occurs at point *B*. This problem can be represented as,

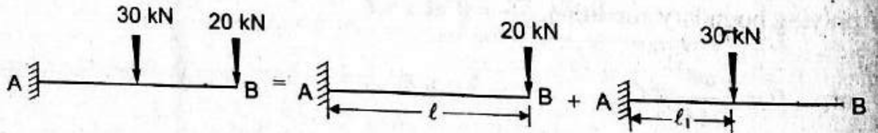


Fig. 2.10

$$\theta_B = \frac{W\ell^2}{2EI} \qquad + \qquad \theta_B = \frac{W_2\ell^2_1}{2EI}$$

$$y_B = \frac{W\ell^3}{3EI} \qquad \qquad y_B = \frac{W_2\ell_1^{\,3}}{3EI} + \frac{W_2\ell_1^{\,3}}{2EI}(\ell - \ell_1)$$

As it is clear from the diagram and expressions, that

$$\theta_B = \left(\frac{W_1\ell^2}{2EI} + \frac{W_2\ell_1^{\,2}}{2EI}\right) = \left(\frac{20\times1000\times2000^2}{2\times2\times10^5\times1.5\times10^8} + \frac{30\times1000\times1000^2}{2.3\times2\times10^5\times1.5\times10^8}\right)$$

$= 0.0018$ rad. <u>Ans</u>

Similarly, deflection at the free end is obtained as

$$y_B = \left(\frac{W_1\ell^3}{3EI}\right) + \left[\frac{W_2\ell_1^{\,3}}{3EI} + \frac{W_2\ell_1^{\,2}}{2EI}(\ell - \ell_1)\right]$$

$$= \left(\frac{20\times1000\times2000^3}{3\times2\times10^5\times1.5\times10^8}\right) + \left[\frac{30\times1000\times1000^3}{3\times2\times10^5\times1.5\times10^8} + \frac{30\times1000\times1000^2}{2\times2\times10^5\times1.5\times10^8}(2000-10)\right]$$

$= 2.61\ mm$ <u>Ans.</u>

Example # 2.2 A cantilever beam of 2.5 *m* span and 120 *mm* ×200 *mm* (b×d) in section is loaded with uniformly distributed load of 165 *kN/m* through its length. Find the maximum deflection. Taking $E = 2\times10^5\ N/mm^2$. If the maximum deflection is to be limited to 8 *mm* what would be the permissible load on the beam?

Sol^n. Given that,

$$w = 16.5\ kN/m = 16500\ N/m$$
$$\ell = 2.5\ m$$
$$E = 2\times10^5\ N/mm^2 = 2\times10^5\times10^{-6} = 2\times10^{11}\ N/m^2$$

Now, $$I = \frac{bd^3}{12} = \frac{0.12\times0.2^3}{12} = 8\times10^{-5} m$$

Maximum deflection would occur at the free end and it is given by

$$y_B = \frac{w\ell^4}{8EI}$$

$$= \frac{16500\times2.5^4}{8\times2\times10^{11}\times8\times10^{-5}} = 0.005m$$

$= 5\ mm$ <u>Ans.</u>

If the allowable deflection is 8 *mm* = 0.008 *m*.,

Then,

$$0.008 = \frac{w \times 2.5^4}{8 \times 2 \times 10^{11} \times 8 \times 10^{-5}} m$$

$w = 26.21\ kN/m$ Ans.

Example # 2.3 A cantilever 2 *m* long is of rectangular section 100 *mm* wide and 200 *mm* deep. It carries a uniformly distributed load of 5 *kN/m* for a length of 1.5 *m* from the fixed end, find the deflection at the free end *B*. Take $E = 10\ GN/m^2$

Sol^n.

We have $y_B = BB' + B'B'' = CC' + B'B''$

CC' = Deflection due to udl for fully loaded span of 1.5 *m*.

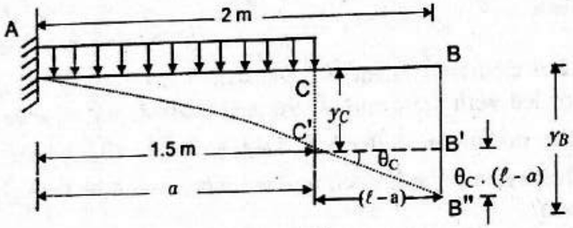


Fig. 2.11

$$CC' = \frac{wa^4}{8EI}$$

$$B'B'' = \theta c.(\ell - a)$$

$$\because (\theta_B = \theta_C)$$

$$= \frac{wa^3}{6EI}(\ell - a)$$

Now,

$\ell = 2\ m$

$b = 100\ mm = 0.1\ m$

$d = 200\ mm = 0.2\ m$

$$I = \frac{bd^3}{12} = \frac{0.1 \times 0.2^3}{12} = 66.66 \times 10^{-6} m^4$$

$w = 5\ kN/m,\ E = 10\ GN/m^2$

$$y_B = \frac{wa^4}{8EI} + \frac{wa^3}{6EI}(\ell - a) = \frac{wa^3}{EI}\left(\frac{a}{8} + \frac{\ell}{6} - \frac{a}{6}\right)$$

$$= \frac{5 \times 1000 \times 1.5^3}{10 \times 10^9 \times 66.66 \times 10^{-6}}\left(\frac{1.5}{8} + \frac{2}{6} - \frac{1.5}{6}\right)$$

$= 6.86\ mm$ Ans.

Example # 2.4 A cantilever, 2 *m* long is of rectangular section 100 *mm* wide and 200 *mm* deep. It carries a uniformly distributed load of 5 *kN/m* for a length of 1.5 *m* from the fixed end; find the deflection at the fixed end. Take $E = 10\ GN/m^2$

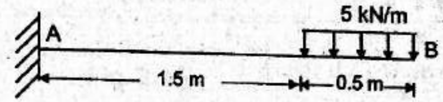


Fig. 2.12

Sol^n. This beam can be treated as a sum of the loading shown below

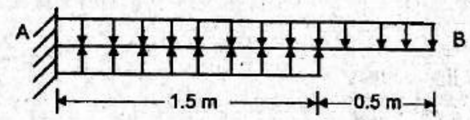


Fig. 2.13

Deflection at $B = -$ deflection due to downward loading + deflection due to upward loading (– ve sign for downward)

$$= -\frac{w\ell^4}{8EI} + \left[\frac{wa^4}{8EI} + \frac{wa^3}{6EI}(\ell - a)\right]$$

$$= -\frac{5 \times 1000 \times 2^4}{8 \times 10 \times 10^9 \times 66.66 \times 10^{-6}} + 6.86\ mm(example2.3)$$

$= -15\ mm + 6.86\ mm$

$= -8.14\ mm$ Ans.

g) Simply supported beams with a point load at the centre

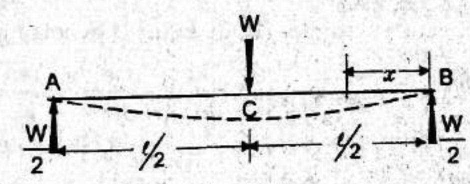


Fig. 2.14

Let a simply supported beam of span ℓ is carrying a central load of *W*. Since the load is symmetrically applied, maximum deflection (y_{max}) will occur at the mid span and each vertical reaction will be equal to *W*/2.
Consider the right half of the span, then,

$$M_x = \frac{W}{2}x \quad \text{or,}\ EI.\frac{d^2y}{dx^2} = \frac{W}{2}.x$$

Integrating,

$$EI.\frac{dy}{dx} = \frac{Wx^2}{4} + C_1 \quad \text{.......(2.28)}$$

(C_1 is integration constant)

$$\frac{dy}{dx} = 0,\ at\ x = \frac{\ell}{2}$$

or, $$0 = \frac{W}{4}\left(\frac{\ell}{2}\right)^2 + C_1$$

$$C_1 = \frac{-W\ell^2}{16}$$

$$\therefore \quad EI.\frac{dy}{dx} = \frac{Wx^2}{4} - \frac{W\ell^2}{16} \quad \text{.......(2.29)}$$

(Equation for slope)

a) <u>Slope at A</u>

Put $x = 0$ in Eq. (2.29), then $\theta_A = \frac{dy}{dx} = -\frac{W\ell^2}{16EI} = \theta_B$ (by symmetry)

Integrating the Eq. (2.29) again,

$$EI.y = \frac{Wx^3}{12} - \frac{W\ell^2.x}{16} + C_2$$

at $x = 0,\ y = 0,\ C_2 = 0$

$$\therefore \quad EI.y = \frac{Wx^3}{12} - \frac{W\ell^2}{16}x \quad \text{.......(2.30)}$$

(Equation for deflection)

b) <u>Deflection at C</u>

Put $x = \frac{\ell}{2}$, in Eq. (2.30), then

$$EI.y_C = \frac{W\left(\frac{\ell}{2}\right)^3}{12} - \frac{W}{16}\ell^2\left(\frac{\ell}{2}\right)$$

$$= \frac{W\ell^3}{96} - \frac{W\ell^3}{32} = -\frac{W\ell^3}{48}$$

$$\therefore y_C = -\frac{W\ell^3}{48EI} \quad \text{.......(2.31)}$$

h) Simply supported beam carrying udl overal whole span

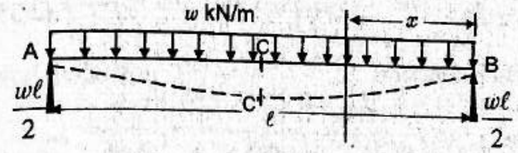


Fig. 2.15

Let a simply supported beam of span ℓ is carrying a uniformly distributed load w over the entire span. The maximum deflection (y_{max}) will occur at mid span and each vertical reaction will be equal to $w.\ell/2$.

Consider a section at a distance x from B

$$M_x = \frac{w\ell}{2}.x - \frac{wx^2}{2}$$

or, $$EI.\frac{d^2y}{dx^2} = \frac{w\ell}{2}.x - \frac{wx^2}{2}$$

Integrating,

$$EI.\frac{dy}{dx} = \frac{w\ell x^2}{4} - \frac{wx^3}{6} + C_1 \quad \text{.......(2.32)}$$

(C_1 is constant of integration)

As the load is symmetrical, the maximum deflection occurs at the mid span when slope is equal to zero (tangent at C' will be horizontal)

i.e. $\frac{dy}{dx} = 0, \quad x = \frac{\ell}{2}$

or, $$0 = \frac{w\ell}{4}\left(\frac{\ell}{2}\right)^2 - \frac{w}{6}\left(\frac{\ell}{2}\right)^3 + C_1$$

$$C_1 = \frac{-w\ell^3}{24}$$

Now, subtituting value of C_1 in Eq. (2.32), we get

$$EI.\frac{dy}{dx} = \frac{w\ell}{4}x^2 - \frac{wx^3}{6} - \frac{w\ell^3}{24} \quad \text{.......(2.33)}$$

(Equation for slope)

a) <u>Slope at A</u>

Put $x = 0$, in the above equation, then $EI.\theta_A = -\frac{w\ell^3}{24}$

$$\therefore \quad \theta_A = -\frac{w\ell^3}{24EI} \quad \text{.......(2.34)}$$

Integrating Eq. (2.33),

$$EI.y = \frac{w\ell x^3}{12} - \frac{wx^4}{24} - \frac{w\ell^3}{24}x + C_2 \quad \text{.......(2.35)}$$

(C_2 is integration constant)

at $x = 0,\ y = 0$ $\quad \therefore C_2 = 0$

$$\therefore \quad EI.y = \frac{w\ell x^3}{12} - \frac{wx^4}{24} - \frac{w\ell^3}{24}x \quad \text{.......(2.36)}$$

This is the equation for deflection.

b) Deflection at mid span, y_{max}

Put, $x = \ell/2$

$$EI.y_{max} = \frac{w\ell}{12}\left(\frac{\ell}{2}\right)^3 - \frac{w}{24}\left(\frac{\ell}{2}\right)^4 - \frac{w\ell^3}{24}\cdot\frac{\ell}{2}$$

$$\text{or,} \quad y_{max} = -\frac{5w\ell^4}{384EI} \quad \text{.......(2.37)}$$

Example # 2.5 A simply supported beam of span 5 m **supports a point load** of 50 kN at its mid span. Calculate slope at ends and the central deflection. Assume $E = 200\ GN/m^2$, $I_{xx} = 15.6\times10^{-6}\ m^4$

Sol^n.

$\ell = 5\ m$

$W = 50\ kN$

$I_{xx} = 15.6\times10^{-6}\ m^4$

$E = 200\times10^9\ \text{N/m}^2$

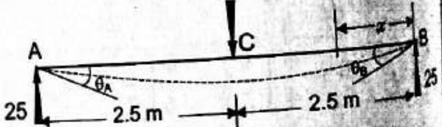


Fig. 2.16

a) Slope at the ends

$$\theta_A = -\frac{W\ell^2}{16EI}$$

$$= -\frac{50\times1000\times5^2}{16\times200\times10^9\times15.6\times10^{-6}}$$

$= -0.025$ radians

$$= -0.025\times\frac{180}{\pi}$$

$= -1.43^0$

Slope at B,

$\theta_B = +1.43 = \theta_A$ (due to load being symmetrical)

b) Central deflection

$$y_{max} = \frac{W\ell^3}{48EI} = \frac{50\times1000\times5^3}{48\times200\times10^9\times15.6\times10^{-6}}\times1000$$

$= 41.7\ mm(\downarrow)$

2.6 Macaulay's Method

In integration method, we usually write an expression for each section of the beam and substitute it in the moment curvature relationship for integration. Problem arises when beam carries a number of loads such that there are different segments and each segment has to be integrated separately. To avoid this problem, Macaulay's method can be used. By this method, it is possible to write only one expression, which is valid for the entire length of the beam. When this expression is integrated, only one constant of integration appears. Thus in certain types of problems, use of this method is very helpful. The method is described below.

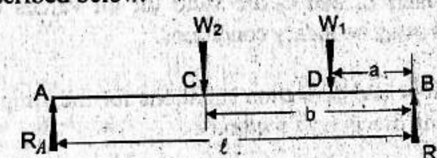


Fig. 2.17

Let AB be a beam of span ℓ loaded with W_1 and W_2 at distances a and b from B respectively

For any section x between D and B,

$$M_x = R_B.x \quad \text{.......(2.38)}$$

Similarly for any section x between C and D,

$$M_x = R_B.x - W_1(x-a) \quad \text{.......(2.39)}$$

and for any section between A and C,

$$M_x = R_B.x - W_1(x-a) - W_2(x-b) \quad \text{.......(2.40)}$$

Thus for any section of the beam, all the above three equations can be combined as

$$M_x = EI\frac{d^2y}{dx^2} = R_B.x \Big| - W_1(x-a) \Big| - W_2(x-b) \quad \text{.......(2.41)}$$

In such a way that for the value of x between

(i) $x=0$ and $x=a$, only the first term of the above equation is considered

(ii) $x=a$ and $x=b$, only the first two terms are considered.

(iii) $x=b$ and $x=\ell$, all the terms are considered.

Now, the Eq. (2.41) is integrated for slope and deflections as shown below.

$$EI\frac{dy}{dx}=R_B\frac{x^2}{2}+C_1 \Big| -\frac{W_1(x-a)^2}{2} \Big| -\frac{W_2(x-b)^2}{2} \quad \text{(for slope)} \ \ldots\ldots \ (2.42)$$

$$EI.y=R_B\frac{x^3}{2\times 3}+C_1x+C_2 \Big| -\frac{W_1}{2}\frac{(x-a)^3}{3} \Big| -\frac{W_2}{2}.\frac{(x-b)^3}{3} \quad \text{(for deflection)} \ \ldots\ldots (2.43)$$

In the above integration procedure, the following points are noted:

(i) The expressions $(x-a)$ and $(x-b)$ are integrated as whole (not as individual terms) and are $\frac{(x-a)^2}{2}$ and $\frac{(x-b)^2}{2}$ respectively.

(ii) The constants C_1 and C_2 are valid for all values of x and are evaluated using boundary conditions.

Example # 2.6 Find slope and deflection equations for the simply supported beam shown below using Macaulay's methods:

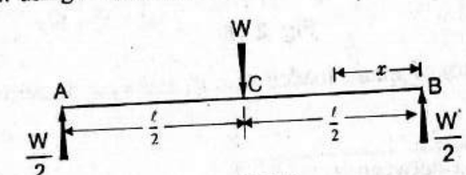


Fig. 2.18

Let us take B as origin. The bending moment at any point in section BC at x from B is

$$M_x=\frac{W}{2}.x \ \ldots\ldots \ (2.44)$$

Similarly for AC,

$$M_x=\frac{W}{2}.x-W(x-\frac{\ell}{2}) \ \ldots\ldots \ (2.45)$$

For the whole beam, combining the above two equations,

$$EI.\frac{d^2y}{dx^2}=\frac{W}{2}.x \Big| -W(x-\frac{\ell}{2}) \ \ldots\ldots \ (2.46)$$

Integrating Eq. (2.46) twice, we get the following two equations.

$$EI\frac{dy}{dx}=\frac{W}{2}.\frac{x^2}{2}+C_1 \Big| -W(x-\frac{\ell}{2})^2.\frac{1}{2} \ \ldots\ldots \ (2.47)$$

$$EI.y=\frac{W}{2}.\frac{1}{2}.\frac{x^3}{3}+C_1x+C_2 \Big| \frac{-W}{2}.\frac{\left(x-\frac{\ell}{2}\right)^3}{3} \ \ldots\ldots (2.48)$$

Now, we know at $x=0, y=0$ and substituting these in Eq. (2.48) we get, $C_2=0$

Also, when $x=\ell$, $y=0$ substituting these, along with $C_2=0$ in Eq. (2.48) we get.

$$0=\frac{W\ell^3}{12}+C_1\ell-\frac{W}{6}\left(\frac{\ell}{2}\right)^3$$

$$C_1\ell=-\frac{W\ell^3}{12}+\frac{W\ell^3}{48}$$

or, $$C_1=-\frac{W\ell^2}{16}$$

Substituting the value of C_1 in Eq. (2.47), we get the slope equation, which is

$$EI\frac{dy}{dx}=\frac{Wx^2}{4}-\frac{W\ell^2}{16} \Big| -\frac{W}{2}\left(x-\frac{\ell}{2}\right)^2 \ \ldots\ldots \ (2.49)$$

As we know that maximum slope occurs at A and B and for its maximum value, $x=0$ and consider Eq. (2.49) up to the dotted line only, we get.

$$EI.\theta_A=-\frac{W\ell^2}{16}$$

or, $$\theta_A=\theta_B=-\frac{W\ell^2}{16EI}$$

Substituting the value of C_1 and C_2 in Eq. (2.48), we get the deflection equation, which is

$$EI.y=\frac{Wx^3}{12}-\frac{W\ell^2x}{16} \Big| -\frac{W}{6}\left(x-\frac{\ell}{2}\right)^3$$

Maximum deflection occurs at C where $x=\ell/2$ and as C lies in BC portion, consider the equation only up to the dotted line.

$$EI\,y_c=\frac{W}{12}\left(\frac{\ell}{2}\right)^3-\frac{W\ell^2}{16}\left(\frac{\ell}{2}\right)=-\frac{W\ell^3}{48}$$

$$\ldots\ldots (2.50)$$

or, $$y_c=\frac{W\ell^3}{48EI} \ \ldots\ldots$$

Example # 2.7 Find maximum slope and deflection for eccentrically loaded beam shown below by using Macaulay's method.

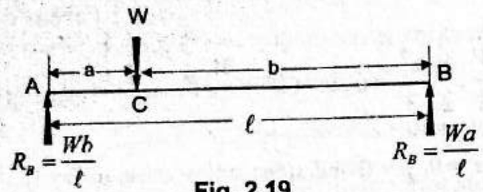


Fig. 2.19

Sol^n. Consider B as origin,

For BC, $M_x = \frac{Wa}{\ell}x$ .... (2.51)

For AC, $M_x = \frac{Wax}{\ell} \Big| -W(x-b)$ .... (2.52)

Combined equation of the above two is

$$M_x = EI\frac{d^2y}{dx^2} = \frac{Wax}{\ell} \Big| -W(x-b) \quad \text{.......... (2.53)}$$

Integrating Eq. (2.53) two times,

$$EI\frac{dy}{dx} = \frac{Wa}{\ell}\cdot\frac{x^2}{2} + C_1 \Big| -W\frac{(x-b)^2}{2} \quad \text{(for slope).......... (2.54)}$$

$$EI\,y = \frac{Wa}{2\ell}\cdot\frac{x^3}{3} + C_1x + C_2 \Big| -\frac{W}{2}\cdot\frac{(x-b)^3}{3} \quad \text{(for delfection)..... (2.55)}$$

at $x=0$, $y=0$, Eq. (2.55) gives $C_2 = 0$
and at $x = \ell$, $y = 0$ gives,

$$0 = \frac{Wa\ell^3}{6\ell} + C_1\ell - W\frac{(\ell-b)^3}{6}, \quad \text{(but } a = \ell - b\text{)}$$

or, $$C_1 = \frac{Wa(a^2-\ell^2)}{6\ell}$$

Substituting values of C_1 in Eq. (2.54) we get the slope equation.

$$EI\frac{dy}{dx} = \frac{Wax^2}{2\ell} + \frac{Wa(a^2-\ell^2)}{6\ell} \Big| -W\frac{(x-b)^2}{2}$$

or, $$EI\frac{dy}{dx} = \frac{Wa}{6\ell}(a^2-\ell^2+3x^2) \Big| -W\frac{(x-b)^2}{2} \quad \text{.............(2.56)}$$

Slope will be maximum at A or B. Substituting $x = 0$ and considering the equation up to dotted line only,

$$EI\,\theta_B = \frac{Wa}{6\ell}(a^2-\ell^2)$$

or, $$\theta_B = \frac{Wa}{6EI\ell}(a^2-\ell^2)$$

Similarly, $\theta_A = \frac{Wb}{6EI\ell}(b^2-\ell^2)$ (substituting b for a)

Substituting values of C_1 and C_2 in Eq. (2.55) and considering the term up to the dotted line only, we get.

$$y = \frac{Wax}{6EI\ell}(a^2-\ell^2+x^2)$$

or, $$y = \frac{Wax}{6EI\ell}(a^2-\ell^2+x^2)$$

As $BC > AC$, maximum deflection occurs somewhere between C and B. At maximum deflection, slope $dy/dx = 0$
So, Eq. (2.56) gives

$$0 = \frac{Wa}{6\ell}(a^2-\ell^2+3x^2)$$

or, $$x = \sqrt{\frac{\ell^2-a^2}{3}}$$

The maximum deflection can be obtained by putting this value of x in deflection equation.

Thus, $$y_{max} = \frac{-Wa}{6EI\ell}\frac{(\ell^2-a^2)^{\frac{1}{2}}}{\sqrt{3}}\left[a^2-\ell^2+\frac{\ell^2-a^2}{3}\right]$$

$$y_{max} = \frac{-Wa}{9\sqrt{3}EI\ell}(\ell^2-a^2)^{\frac{3}{2}}$$

Example # 2.8 A simply supported girder of span 14 m carries two point loads of 1200 and 800 kN as shown. Calculate the deflection of the girder at points under the two loads using Macaulay's method. Take $I = 16 \times 10^3\ cm^4$ and $E = 2.1 \times 10^4\ kN/cm^2$.

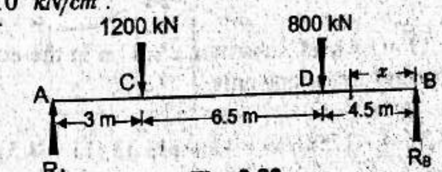


Fig. 2.20

Sol^n. Taking moment about A,

$$R_B \times 14 = 1200 \times 3 + 800 \times 9.5$$

or $$R_B = 800\ kN,\ R_A = 1200 \quad (R_A + R_B = 1200+800)$$

B.M. at any section X from B is

$$M_x = EI\frac{d^2y}{dx^2} = 800x \;\Big|\; -800\,(x-4.5) \;\Big|\; -1200\,(x-11) \quad \ldots\ldots\ldots\ldots\ldots(2.57)$$

Integrating the equations,

$$EI\frac{dy}{dx} = 800.\frac{x^2}{2} + C_1 \;\Big|\; -800\frac{(x-4.5)^2}{2} \;\Big|\; -1200\frac{(x-11)^2}{2} \quad \ldots\ldots\ldots\ldots(2.58)$$

Integrating this once again,

$$EI.y = 400\frac{x^3}{3} + C_1x + C_2 \;\Big|\; -400\frac{(x-4.5)^3}{3} \;\Big|\; -600\frac{(x-11)^3}{3} \quad \ldots\ldots\ldots(2.59)$$

We know, when $x = 0,\ y = 0,\ \therefore C_2 = 0$

and at $x = 14 \quad y = 0$ and substituting this value in Eq. (2.59)

$$\text{or, } 0 = 133.33 \times 14^3 + C_1 \times 14 + 0 - 133.33\,(14 - 4.5)^3 - 200(14-11)^3$$

$$\text{or, } C_1 = \frac{-133.33\times14^3 + 133.33(14-4.5)^3 + 200(14-11)^3}{14} = -17581.69$$

Substituting the values in Eq. (2.59),

$$EI.y = 133.33\,x^3 - 17581.69\,x \;\Big|\; -133.33(x-4.5)^3 \;\Big|\; -200(x-11)^3 \quad \ldots\ldots\ldots(2.60)$$

For the deflection under 800 kN load, substituting $x = 4.5$ in the equation and take the terms up to first dotted line only.

$$\text{or,} \quad y_D = \frac{133.33\times4.5^3 - 17581.69\times4.5}{EI}$$

$$= \frac{66968}{2.1\times10^4\times10^4\times16\times10^3\times10^{-8}}$$

$$= 1.99\,cm \text{ (Downward)}$$

For defection under 1200 kN load, substitute $x = 11m$ in the equation and take the terms up to the second dotted line only.

$$\text{or,} \quad EI.y = 133.33\,x^3 - 17581.69 \times 11 - 133.33\,(11 - 4.5)^3$$

$$\text{or,} \quad y = \frac{133.33\times11^3 - 17581.69\times11 - 133.33(11-4.5)^3}{2.1\times10^4\times10^4\times16\times10^3\times10^{-8}}$$

$$= 1.56\ cm \text{ (Downward)}$$

2.7 MOMENT AREA METHOD

The integration method described earlier allows us to form general equation for slope and defections for the entire structure. Such equation e.g. $EI.y = -\ Wx^3/6 + W\ell^2\,x/2 + W\ell^2/3$ can provide deflection of a beam at any desired sections (centre, quarter of spans ... etc) by substituting the value of x as the equation is valid for the entire structure. It may not always be necessary to find such general equation specifically when the parameters (slope, deflection) are desired only at some specific sections. Moment area method is one of the most convenient methods to use in such case. It however makes use of the governing differential equation in convention with the geometry of the elastic curve. This method considers deflection due to only flexure and not due to axial force and shear. The derivation of moment area theorem is given below.

Let: AB be the segment of the beam whose elastic curve after application of load is represented $AmnB$. Draw tangents T_A and T_B from A and B respectively. T_A intersects the vertical line through B at C and $BC = d$. The slopes at A and B are θ_A and θ_B. θ_{AB} represents the change in angle between the tangents T_A and T_B.

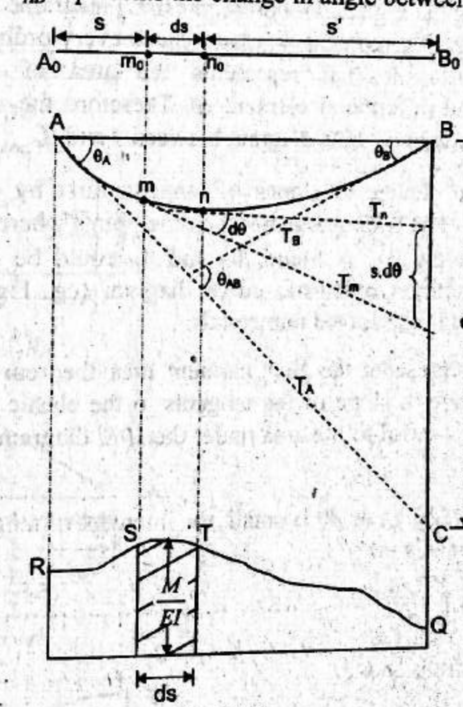


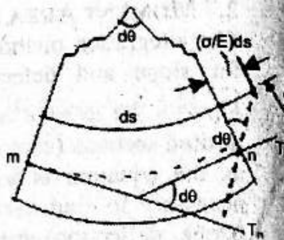
Fig. 2.21

Consider the differential element mn and draw the tangents T_m and T_n from the points m and n respectively. The change in angle between these tangents $d\theta$ is obtained by simple geometry as shown below.

$$d\theta = \frac{perp}{base} = \frac{\frac{\sigma}{E}ds}{y} \quad (put\ \sigma = \frac{M}{I}.y), \text{ then,}$$

$$= \left(\frac{M}{I}.y\right)\frac{ds}{E}.\frac{1}{y}$$

$$= \frac{M}{EI}.ds$$



Now, the change in the angle between the tangent T_A and T_B is obtained by integrating $d\theta$ with limit from A to B

i.e. $\theta_{AB} = \int d\theta = \int_A^B \frac{M}{EI}.ds$ .... (2.61)

$\therefore \quad \theta_{AB} = A,$

where A = Area of $\frac{M}{EI}$ diagram

The lower portion of the Fig.(2.21) represents (*M/EI*) diagrams of the beam. It is essentially the bending moment diagram whose every ordinate is divided by *EI*. The product (*M/EI*).*ds* represents the area of *M/EI* diagram corresponding to the differential element *ds*. Therefore the integral of Eq. (2.61) represents the area of *M/EI* diagram between *A* and *B*.

Thus one may find change in slopes of tangents first by computing the corresponding area of *M/EI* diagrams between the points where tangents were drawn. Once the angle θ_{AB} is found, θ_A and θ_B could be determined by geometrical consideration of elastic curve diagram (eg. Fig. 2.21 middle porion) as illustrated in the solved numericals.

Eq. (2.61) above represents the first moment area theorem which can be stated as "The change in slope of the tangents to the elastic curve between two points *A* and *B* is equal to the area under the *M/EI* diagram between these two points."

Again referring Fig. (2.21), as $d\theta$ is small, the intercept of tangents T_m and T_n on *BC* can be written as $s'd\theta$

Obviously, $\quad d = \int_B^A s'\,d\theta$ .... (2.62)

but, $d\theta = \frac{M}{EI}ds$ from above.

Substituting this in Eq. (2.62)

or, $\quad d = \int_A^B \frac{M}{EI}.s'\,ds \Rightarrow \frac{M}{EI}.ds = Area,\ \ s' = \text{distance of C.G. } i.e. = \bar{x}$

$\therefore \quad d = A\bar{x}$

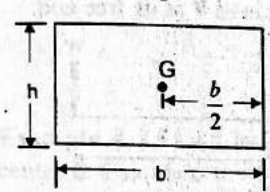
The deflection *d* of the tangent at *A* from point *B* is thus equal to the moment of *M/EI* diagram about *B*, which is the second moment area theorem and can be expressed as

"The deflection of point *B* on the elastic curve from the tangent to this curve at point *A* is equal to static moment about an axis through *B* of the area under the *M/EI* diagram between points *A* and *B*."

It is noted that the area of the *M/EI* diagram above the reference line (zero line) will be taken as positive and vice-versa.

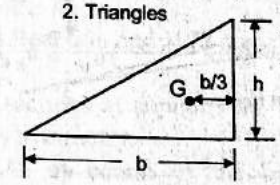
In using this method to find deflection of structures, we often require computing area and center of gravity of various diagrams. Below given are figures, which we usually come across.

1. Rectangle

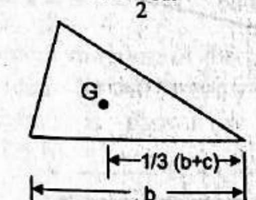


A = b.h

2. Triangles

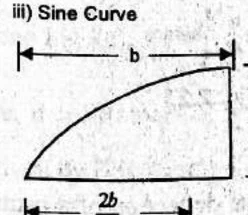


$A = \frac{1}{2}b.h$



$A = \frac{1}{2}b.h$

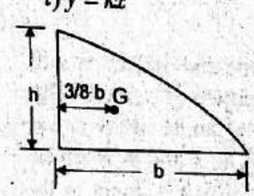
iii) Sine Curve



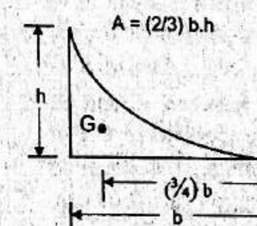
$A = \frac{2bh}{\pi}$

3. Curves

i) $y = kx^2$

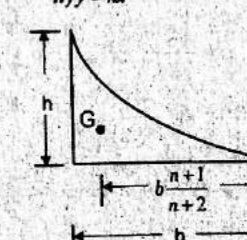


A = (2/3) b.h



A = (1/3). b.h

ii) $y = kx^n$



$A = \frac{1}{n+1}bh$

Parabolic curves

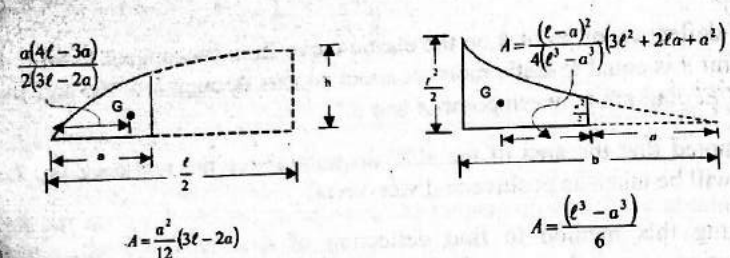


Fig. 2.22

Example # 2.9 Using moment area method, calculate the maximum slope and defection of a cantilever beam loaded with point load W at its free end.

Sol^n.

The slope and deflection will be maximum at point B. The change of angle between the tangents at A and B is given by θ_{AB}, which is equal to θ_B as this is a case of cantilever. So, $\theta_{max} = \theta_B$ = Area of M/EI diagram between A and B. Similarly, the intercept d of the tangent at A is equal to the deflection at B. Thus, $y_{max} = y_B = d = M/EI$ diagram between A and B and about B.

$$\therefore \theta_{max} = \theta_B = \frac{1}{2} \times \ell \times \frac{w\ell}{EI}$$

$$= \frac{w\ell^2}{2EI} \quad \underline{\text{Ans.}}$$

$$y_{max} = y_B = d = A\bar{x}$$

$$= \frac{1}{2} \times \ell \times \frac{W\ell}{EI} \cdot \frac{2}{3} \cdot \ell = \frac{W\ell^2}{3EI} \quad \underline{\text{Ans.}}$$

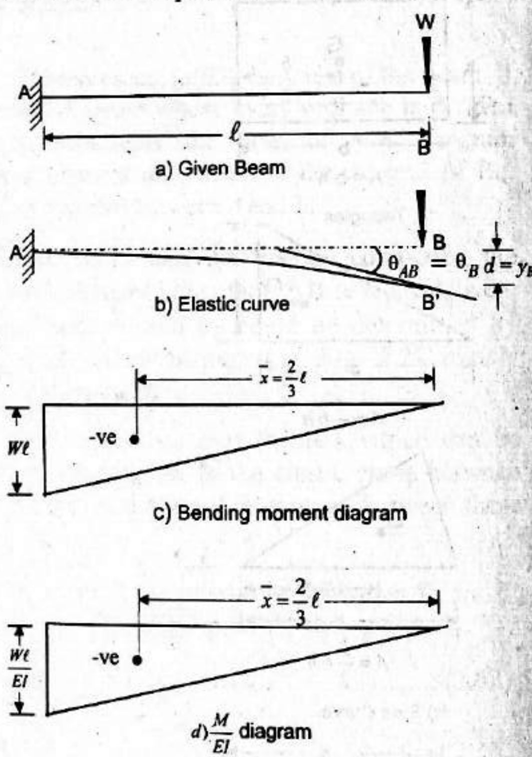


Fig. 2.23

Example # 2.10 Calculate the maximum slope and deflection of a cantilever beam loaded with uniformly distributed load w along the entire span.

Sol^n.

As in the previous example,

$$\theta_{max} = \theta_B = A = \frac{1}{3} \cdot \ell \cdot \frac{w\ell^2}{2EI}$$

$$= \frac{w\ell^3}{6EI} \quad \underline{\text{Ans.}}$$

$$y_{max} = y_B = d = A\bar{x} = \frac{w\ell^3}{6EI} \cdot \frac{3}{4}\ell$$

$$= \frac{w\ell^4}{8EI} \quad \underline{\text{Ans.}}$$

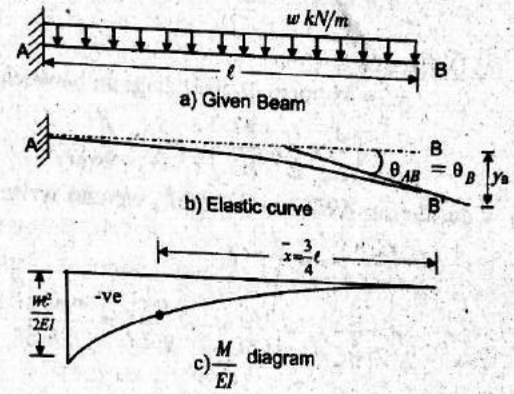


Fig. 2.24

Example # 2.11 AB be a beam which is loaded with a point load W at the center C. Calculate slope at A and deflection at C of the beam.

Sol^n.

a) <u>Reaction and M/EI diagram</u>

Reactions at supports are $W/2$ by symmetry.

$B.M.$ at centre of the beam $= \frac{W}{2} \times \frac{\ell}{2} = \frac{W\ell}{4}$ and bending moment diagram of the beam divided by EI i.e. M/EI diagram is shown in Fig. (2.7.5 b)

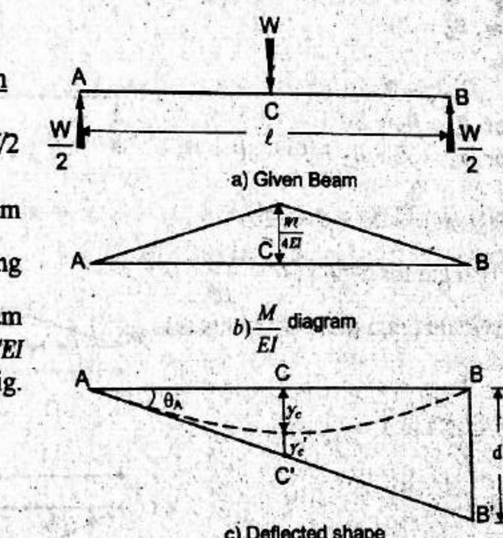


Fig. 2.25

b) <u>Slope at A</u>

Refering Fig. (c), $\theta_A = \frac{d}{\ell}$

Where, d = Moment of Area of $\frac{M}{EI}$ Diagram between A and B and about B.

(As given by II theorem)

$$= \left(\frac{1}{2} \times \ell \times \frac{W\ell}{4EI}\right)\frac{\ell}{2} = \frac{W\ell^3}{16EI}$$

$$\therefore \theta_A = \frac{W\ell^3}{16EI \times \ell} = \frac{W\ell^2}{16EI} \quad \underline{\text{Ans.}}$$

c) <u>Deflection at C</u>

$y_c' =$ Moment of M/EI diagram between A and C and about C

$$= \left(\frac{1}{2} \times \frac{\ell}{2} \times \frac{W\ell}{4EI}\right) \frac{\ell}{2} \times \frac{1}{3} = \frac{W\ell^3}{96EI}$$

Considering $\Delta ACC'$ and $\Delta ABB'$, we can write

$$\frac{y_c + y_c'}{\ell/2} = \frac{d}{\ell}$$

$$\text{or,} \quad y_c = \frac{d}{2} - y_c' = \frac{W\ell^3}{16EI} \times \frac{1}{2} - \frac{W\ell^3}{96EI} = \frac{3W\ell^3 - W\ell^3}{96EI} = \frac{W\ell^3}{48EI} \quad \underline{\text{Ans.}}$$

Example # 2.12 Using the moment area theorem compute the slope of the elastic curve at points A and C and the deflection at C.

Sol^n.

a) <u>Reaction and M/EI diagram</u>

$\Sigma M_A = 0$

or, $R_B \times 18 = 30 \times 12$

or, $R_B = 20$

$\Sigma F_y = 0$

or, $R_A + R_B = 30$

or, $R_A = 30 - R_B = 30 - 20 = 10$

B.M. at $D = 20 \times 6 = 120$

Ordinate for M/EI at $D = \dfrac{120}{EI}$

M/EI diagram is shown in Fig.(c)

b) <u>Slope at A</u>

$$\theta_A = \frac{d}{\ell} = \frac{d}{18}$$

$d =$ Moment of area of M/EI diagram between A and B and about B

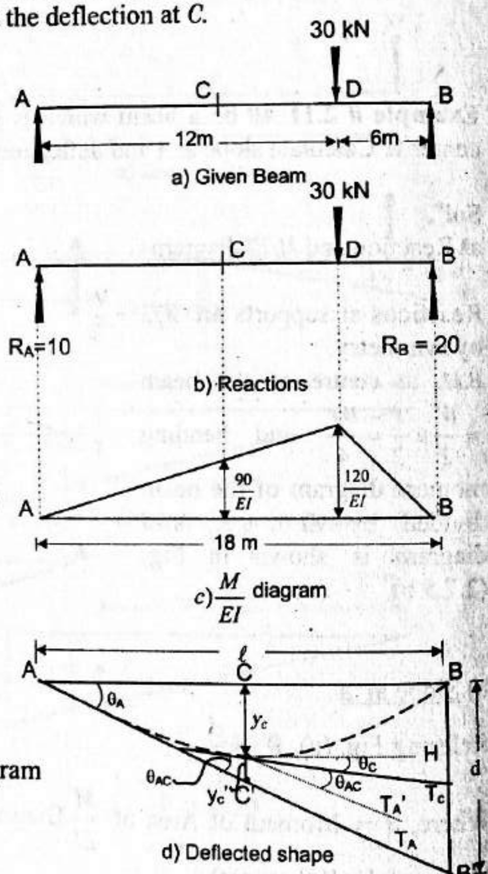


Fig. 2.26

$$= A\bar{x} = \frac{1}{EI}\left[\left(\frac{1}{2} \times 6 \times 120\right)\left(\frac{2}{3} \times 6\right) + \left(\frac{1}{2} \times 12 \times 120\right)\left(6 + \frac{12}{3}\right)\right]$$

$$d = \frac{8640}{EI}$$

$$\text{or,} \quad \theta_A = \frac{8640}{EI \times 18} = \frac{480}{EI}$$

c) <u>Slope at C</u>

$\theta_{AC} =$ Area of M/EI diagram between A and C

$$= \frac{1}{2} \times 9 \times \frac{90}{EI} = \frac{405}{EI}$$

The dotted lines H and T_A' are drawn which are parallel to the horizontal (AB) and the tangent T_A respectively.

$$\text{Now,} \quad \theta_c = \theta_A - \theta_{AC} = \frac{480}{EI} - \frac{405}{EI} = \frac{75}{EI} \quad \underline{\text{Ans.}}$$

d) <u>Deflection at C</u>

$y_c' =$ Moment of area of M/EI diagram between A and C and about C.

$$= \left(\frac{1}{2} \times 9 \times \frac{90}{EI}\right)\left(\frac{9}{3}\right) = \frac{1215}{EI}$$

Again we have,

$$\frac{y_c + y_c'}{9} = \theta_A = \frac{480}{EI} \quad \text{that gives,} \quad y_c = \frac{3015}{EI} \quad \underline{\text{Ans.}}$$

Example # 2.13 Determine slope at A and C and deflection at E of the beam using moment area method.

Sol^n. Reaction, M/EI diagram and deflected shape of the beam are given below

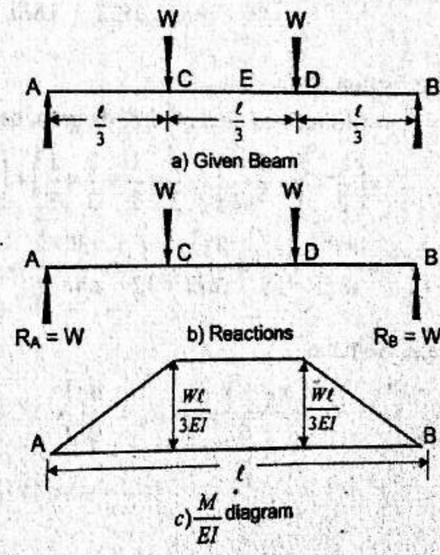
The value of reactions

$R_A = R_B = W$ symmetry,

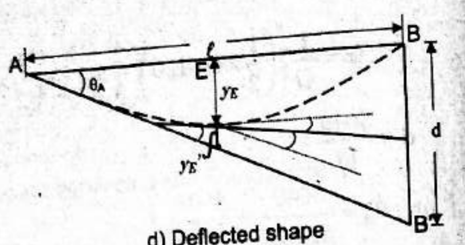
B.M. at $D = R_A \times \dfrac{\ell}{3} = \dfrac{w\ell}{3}$

Ordinate of M/EI at $D = \dfrac{W\ell}{3EI}$

B.M. at $C = R_B \times \dfrac{\ell}{3} = \dfrac{W\ell}{3}$



Ordinate of $\dfrac{M}{EI}=\dfrac{W\ell}{3EI}$



d) Deflected shape

Fig. 2.27

a) Slope at A

$$\theta_A=\frac{d}{\ell}$$

d = Moment of area of *M/EI* diagram between *A* and *B* about *B*, which is,

$$=A\bar{x}=\left(\frac{1}{2}\times\frac{\ell}{3}\times\frac{W\ell}{3EI}\right)\left(\frac{2\ell}{3}+\frac{\ell}{3}\times\frac{1}{3}\right)+\left(\frac{\ell}{3}\times\frac{W\ell}{3EI}\right)\left(\frac{\ell}{2}\right)+\left(\frac{1}{2}\times\frac{\ell}{3}\times\frac{W\ell}{3EI}\right)\left(\frac{\ell}{3}\times\frac{2}{3}\right)$$

$$=\frac{W\ell^2}{18EI}\times\frac{7\ell}{9}+\frac{W\ell^2}{9EI}\times\frac{\ell}{2}+\frac{W\ell^2}{18EI}\times\frac{2\ell}{9}$$

$$=\frac{W\ell^3}{18EI}\left(\frac{7\ell+9\ell+2\ell}{9}\right)=\frac{W\ell^3}{9EI}$$

$$\theta_A=\frac{W\ell^2}{9EI}\times\frac{1}{\ell}=\frac{W\ell^2}{9EI}\ \underline{\text{Ans}}$$

b) Slope at C

θ_{AC} = Area of *M/EI* diagram between *A* and *C*,

$$=\frac{1}{2}\times\frac{\ell}{3}\times\frac{W\ell}{3EI}=\frac{W\ell^2}{18EI}$$

As done in example 2.12,

$$\theta_c=\theta_A-\theta_{AC}=\frac{W\ell^2}{9EI}-\frac{W\ell^2}{18EI}=\frac{W\ell^2}{18EI}\ \underline{\text{Ans.}}$$

c) Deflection at E

y_E' = Moment of area of *M/EI* diagram between *A* and *E* and about *E*,

$$=\left(\frac{1}{2}\times\frac{\ell}{3}\times\frac{W\ell}{3EI}\right)\times\left(\frac{\ell}{3}\times\frac{1}{2}+\frac{\ell}{3}\times\frac{1}{3}\right)+\left(\frac{\ell}{3}\times\frac{1}{2}\times\frac{W\ell}{3EI}\right)\times\left(\frac{\ell}{3}\times\frac{1}{2}\times\frac{1}{2}\right)$$

$$=\frac{W\ell^2}{18EI}\times\frac{5\ell}{18}+\frac{W\ell^2}{18EI}\times\frac{\ell}{12}=\frac{13W\ell^3}{648EI}$$

Again, we have,

$$\frac{y_E+y_E'}{\frac{\ell}{2}}=\theta_A=\frac{W\ell^2}{9EI}$$

$$\text{or, } y_E=\frac{W\ell^3}{18EI}-y_E'=\frac{W\ell^3}{18EI}-\frac{13W\ell^3}{648EI}=\frac{23}{648}\cdot\frac{W\ell^3}{EI}\ \underline{\text{Ans.}}$$

Example # 2.14 A simple beam *AB* supports two concentrated loads of 100 *kN* and 200 *kN* as shown in the figure. Calculate the maximum δ_{max} of the beam, assuming $E=200\ GPa$ and $I=1.2\times10^9\ mm^4$.

Sol^n.

Reaction and *M/EI* diagram

$\sum M_A=0$

or, $R_B\times10=200\times7.5+100\times5$

or, $R_B=200$

$\sum F_y=0$,

$R_A+R_B=100+200$

or $R_A=300-200=100$

B.M. at $D=R_B\times2.5$

$=\ 200\times2.5=500$,

$=$ Ordinate of $\dfrac{M}{EI}=\dfrac{500}{EI}$

BM at $C=R_A\times5=100\times5=500$

Ordinate of $\dfrac{M}{EI}=\dfrac{500}{EI}$

M/EI diagram is shown in Fig. (c)

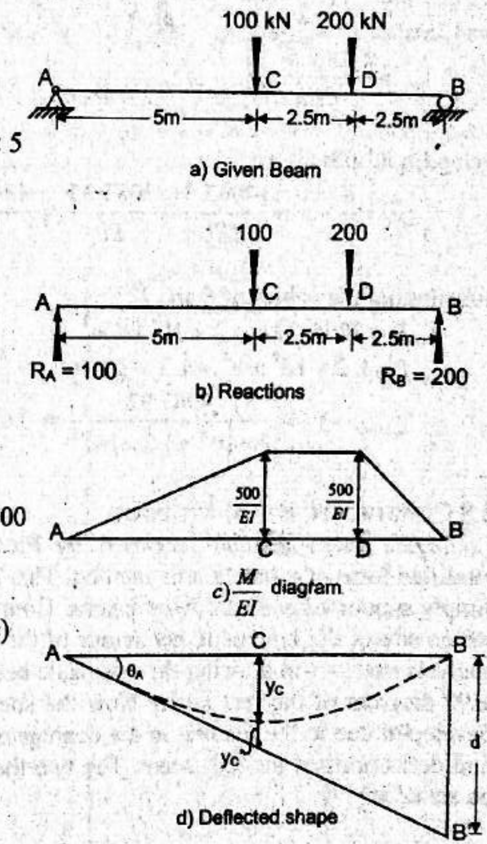


Fig. 2.28

a) Maximum deflection

As seen from *M/EI* diagram, deflection will be maximum at *C* or at *D*. (ordinate are equal), From the Fig. (d), we have,

$$\frac{y_c+y_c'}{\frac{\ell}{2}}=\frac{d}{\ell}\ \text{.......................................}\ (2.63)$$

Where d = moment of area of *M/EI* diagram between *A* and *B* and about *B*.

Moment of M/EI diagram between A and C and about C.

Computing these unknown quantities

$$d = \frac{1}{EI}\left[\left(\frac{1}{2}\times\frac{10}{2}\times 500\right)\left(5+\frac{5}{3}\right)+(2.5\times 500)\left(2.5+\frac{2.5}{2}\right)+\left(\frac{1}{2}\times 2.5\times 500\right)\left(\frac{2}{3}\times 2.5\right)\right]$$

$$=\frac{14062.5}{EI}$$

Using Eq. (2.63) above,

$$y_c = \frac{d}{2} - y' = \frac{14062.5}{2EI} - \frac{2083.33}{EI} = \frac{4947.92}{EI}$$

Substituting the values of E and I,

$E = 2006\ GPa = 2\times 10^8\ kN/m^2$

$I = 1.2\times 10^9\ mm^4 = 1.2\times 10^{-3}\ m^4$

$$y_{max} = y_c = \frac{4947.92}{2\times 10^8\times 1.2\times 10^{-3}} = 0.0206\,m = 20.61\ mm$$ Ans.

2.8 Conjugate Beam Method

Conjugate beam method proposed by Prof. H.F.B Muller–Breslau is a modified form of moment area method. This method can directly be used for simply supported and cantilever beams. Conjugate beam is a fictitious beam which shows the kinematic behaviour of the actual beam. The procedure of analysis involves in drawing the conjugate beam first which is then loaded by *M/EI* diagram of the real beam. Now the shear force and bending moments developed due to the loading in the conjugate beam corresponds to the slope and deflections of the real beam. The two theorems of this method can thus be stated as:

Theorem 1:

The shear at any point on the conjugate beam is equal (in sign and value) to the slope at the corresponding point on the real beam.

Theorem 2:

The moment at any point on the conjugate beam is equal (in sign and value) to the deflection at the corresponding point on the real beam.

Relationship Between Real Beam and its Conjugate Beam:

(i) The span of the real and the conjugate beams are equal.

(ii) The *M/EI* diagram of the real beam becomes the load diagram of the conjugate beam. (sign convention used for moment area theorems may be applied)

(iii) The shear at any section of the conjugate beam is equal to the slope of the real beam.

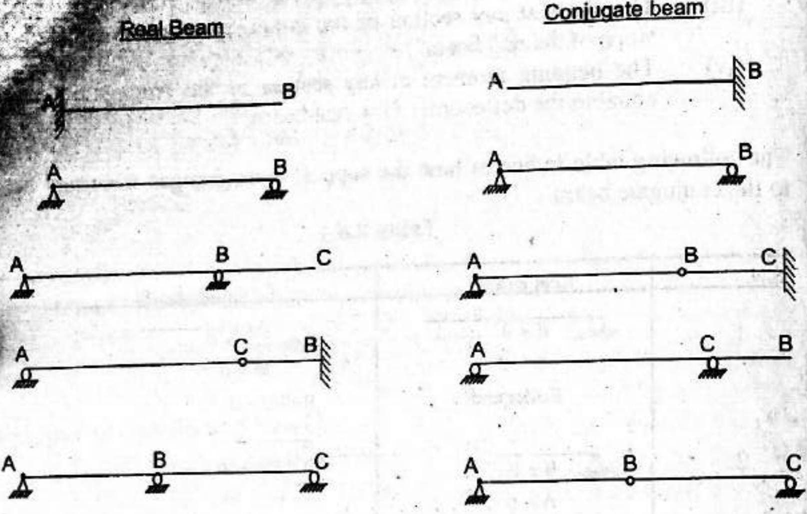
(iv) The bending moment at any section of the conjugate beam is equal to the deflection of the real beam.

The following table indicates how the support ends changes from real beam to the conjugate beam.

Table 2.8.1

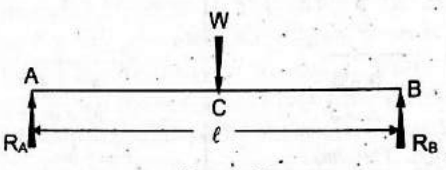
S.N.	Real Beam	Conjugate Beam
1.	$\theta \neq 0$, $\Delta = 0$ Roller end	$S.F. \neq 0$, $M = 0$ Roller end
2.	$\theta \neq 0$, $\Delta = 0$ Hinged end	$S.F. \neq 0$, $M = 0$ Hinged end
3.	$\theta = 0$, $\Delta = 0$ Fixed end	$S.F. = 0$, $M = 0$ Free end
4.	$\theta \neq 0$, $\Delta \neq 0$ Free end	$S.F. \neq 0$, $M \neq 0$ Fixed end
5.	$\theta \neq 0$, $\Delta = 0$ Interior support	$S.F. \neq 0$, $M = 0$ Interior hinge
	$\theta \neq 0$, $\Delta \neq 0$ Interior hinge	$S.F. \neq 0$, $M \neq 0$ Interior support

The figures given below shows the examples of conversion of real beam into conjugate beam.



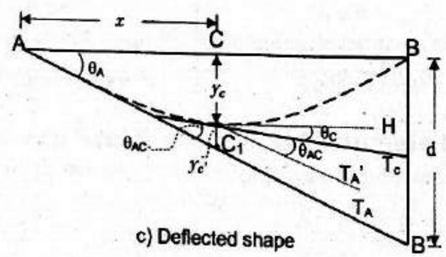
Prove of conjugate beam theorems

Let us consider a simply supported beam loaded with a point load W at the centre of the span. Its M/EI diagram, and deflected shapes are shown in Fig. (2.29).

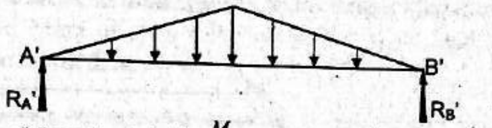


a) Loaded Beam

b) $\frac{M}{EI}$ diagram



c) Deflected shape



d) Beam loaded with $\frac{M}{EI}$ diagram, (Conjugate beam)

Fig. 2.29

From Fig (c), we have

$$\theta_A = \frac{BB'}{\ell} = \frac{1}{\ell} \times \text{moment of } M/EI \text{ diagram between } A \text{ and } B \text{ and about } A.$$

The quantity at right hand side (moment of area divided by span) represents the reaction at A, R_A of the conjugate beam loaded with M/EI. This is explained in the note with the Fig. (2.30).

$$\therefore\ \theta_A = R_A'$$

Explanation

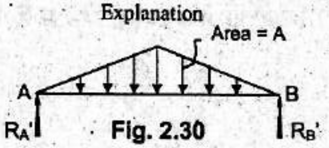


Fig. 2.30

$$\sum M_B = 0,$$

$$R_A' \times \ell = A\,\bar{x}$$

$$\therefore\ R_A' = \frac{\text{Moment of area}}{\ell}$$

Again from Fig. (c)

θ_{AC} = Area of M/EI diagram between A and C

= Applied load left to the section C

The slope at C is now given by

$$\theta_c = \theta_A - \theta_{AC}\ ;$$

$= R_A'$ – Applied load left to C.

= Shear force at C of conjugate beam.

This proves the first theorem.

Similarly, from Fig. (c)

$$y_c = CC_1 - y_c'$$

y_c' = Moment of M/EI diagram between A and C and about C.

$$CC_1 = x.\theta_A = x.R_A'$$

$\therefore\quad y_c = x.R_A'$ – Moment of M/EI both A and C about C.

= Bending moment at C

This proves the second theorem.

Example # 2.15 Using Conjugate beam method, determine slope and deflection at B of the cantilever beam loaded with a point load W at its end.

Sol^n. Fig. (a), (b) and (c) shows the real beam, its bending moment diagram and the conjugate beam respectively. The bending moment ordinates divided by EI (M/EI diagram) become loading for the conjugate beam $A'B'$. According to the conjugate beam theorems, bending moment at B' of the

conjugate beam would represent the deflection at B whereas shear force represents the slope.

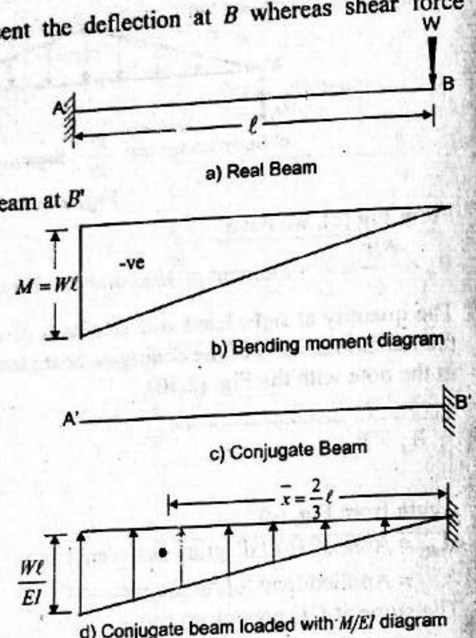


Fig. 2.31

Shear force (SF) of conjugate beam at B'

$$=\frac{1}{2}\times\ell\times\frac{W\ell}{EI}=\frac{W\ell^2}{2EI},$$

$$\therefore\theta_B=-\frac{W\ell^2}{2EI}$$

Bending moment (BM.) at B'

$$=\frac{1}{2}\times\ell\times\frac{W\ell}{EI}\times\frac{2}{3}.\ell$$

$$\therefore\quad y_B=-\frac{W\ell^3}{3EI}\ \underline{\text{Ans.}}$$

Example # 2.16 Calculate maximum slope and deflection of the cantilever beam of span ℓ loaded with uniformly distributed load w. Use conjugate beam method.

Sol^n. Fig. (c) shows the conjugate beam loaded with M/EI diagram of the real beam.

Obviously, for a cantilever beam, slope and deflection are maximum at end B.

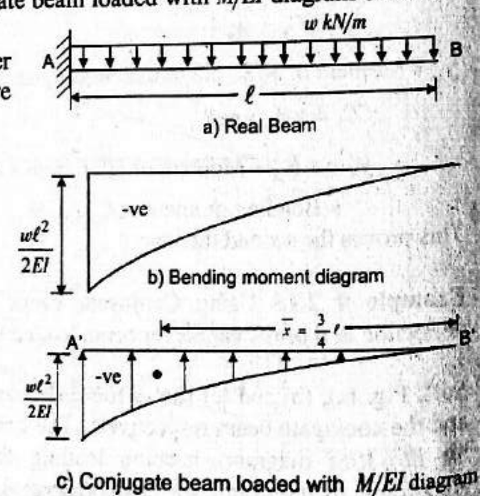


Fig. 2.32

SF of conjugate beam at B' is

$$=\frac{1}{3}\times\frac{w\ell^2}{2EI}.\ell=\frac{w\ell^3}{6EI}$$

$$\therefore\theta_B=-\frac{w\ell^3}{6EI}\ \underline{\text{Ans.}}$$

BM of conjugate beam at B'

$$=\frac{1}{3}\frac{w\ell^2}{2EI}.\ell\times\frac{3\ell}{4}=\frac{w\ell^4}{8EI}$$

$$\therefore\ y_B=-\frac{w\ell^4}{8EI}\ \underline{\text{Ans.}}$$

Example # 2.17 Calculate the slope at ends and the deflection at the centre of simply supported beam of span ℓ loaded with a point load W at the centre. Use conjugate beam method.

Sol^n. The real beam, its bending moment diagram and the conjugate beam loaded with M/EI diagram are shown in the Fig. (2.33). To find the shear force and bending moment in the conjugate beam, we need to find the reactions R_A' and R_B' first.

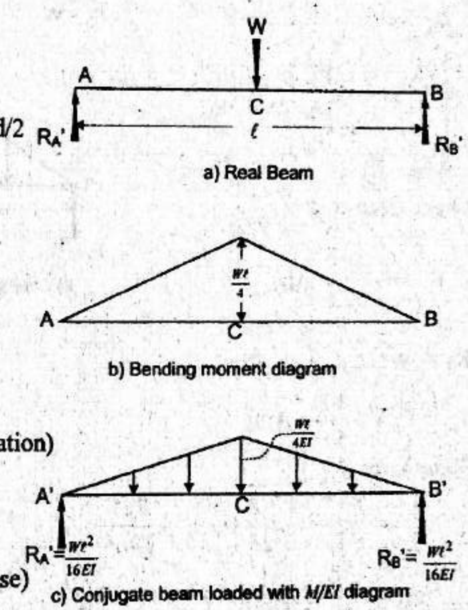


Fig. 2.33

By symmetry of loading,

$R_A' = R_B'$ = Total downward load/2

$$=\left(\frac{1}{2}\times\ell\times\frac{W\ell}{4EI}\right)\times\frac{1}{2}$$

$$=\frac{W\ell^2}{16EI}$$

Now, shear force at A'

$$=R_A'=\frac{W\ell^2}{16EI}$$

$$\therefore\ \theta_A=-\frac{W\ell^2}{16EI}\ \text{(Clockwise rotation)}$$

By symmetry

$$\theta_A=\theta_B\ =+\frac{W\ell^2}{16EI}\text{(Anticlockwise)}$$

Bending moment at C

$$=\left(\frac{W\ell^2}{16EI}\right)\times\frac{\ell}{2}-\left(\frac{1}{2}\times\frac{\ell}{2}\times\frac{W\ell}{4EI}\right)\times\left(\frac{\ell}{2}\times\frac{1}{3}\right)=\frac{W\ell^3}{32EI}-\frac{W\ell^3}{96EI}=\frac{W\ell^3}{48EI}\ \therefore$$

$$y_c=+\frac{W\ell^3}{48EI}\ \text{(Downward deflection)}$$

Example # 2.18 Using conjugate beam method; determine slope at A, B and C and deflection at E of the beam shown below.

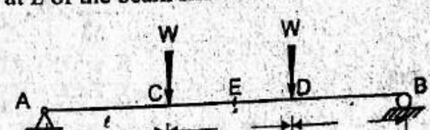


Fig. 2.34

Sol^n^. Due to the symmetry, the reaction in the beam will be half of the total vertical load in each of the support. The bending moment diagram and the conjugate beam loaded with *M/EI* diagram is shown in Fig. (2.35).

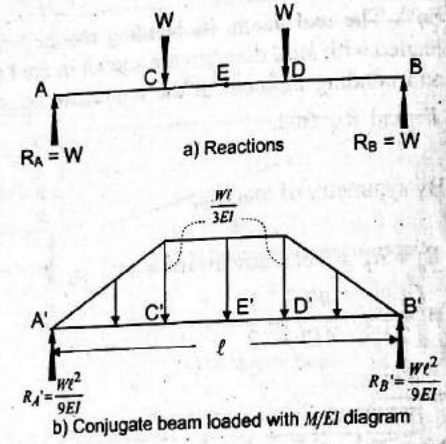


Fig. 2.35

Reaction of the conjugate beam

$R_A' = R_B' =$ Total load/2

(By symmetry)

$$\left[\frac{\left(\frac{\ell}{3}+\ell\right)}{2}\times\frac{W\ell}{3EI}\right]\frac{1}{2}=\frac{W\ell^2}{9EI}$$

Shear force at $A' = \dfrac{W\ell^2}{9EI}$

$\therefore \theta_A = -\dfrac{W\ell^2}{9EI}$ Ans.

By symmetry, $\theta_A = -\theta_B$

$\therefore \theta_B = +\dfrac{W\ell^2}{9EI}$ Ans.

Shear force at C

$$=\frac{W\ell^2}{9EI}-\frac{1}{2}\times\frac{\ell}{3}\times\frac{W\ell}{3EI}=\frac{W\ell^2}{18EI}$$

$\therefore \theta_c = -\dfrac{W\ell^2}{18EI}$ Ans.

Bending moment at

$$E' = \frac{W\ell^2}{9EI}\times\frac{\ell}{2}-\left[\left\{\frac{\ell}{6}\times\frac{W\ell}{3EI}\times\frac{\ell}{6\times2}+\frac{1}{2}\times\frac{\ell}{3}\times\frac{W\ell}{3EI}\left(\frac{\ell}{6}+\frac{1}{3}\cdot\frac{\ell}{3}\right)\right\}\right]$$

$$=\frac{W\ell^3}{18EI}-\left[\frac{W\ell^3}{216EI}+\frac{5W\ell^3}{324EI}\right]$$

$$=\frac{23}{648}\frac{W\ell^3}{EI}\ \text{Ans.}$$

$\therefore y'_E = -\dfrac{23}{648}\dfrac{W\ell^3}{EI}$ Ans.

Example # 2.19 An Indian Standard Median Weight Beam (ISMB) 250 × 125 *mm* carries a concentrated load of 2 tonnes at point *C*, 6 *m* the left end of a beam of span 9 *m*. Find deflection of the beam under the load and the maximum deflection on the span. Take $I_{xx} = 5131.6\ cm^4$ and $E = 2\times10^4\ kN/cm^2$ and use conjugate beam method.

Sol^n^. The real beam, bending moment diagram and the conjugate beam loaded with *M/EI* diagram is shown in Fig. The ordinate of *B.M.* diagram is calculated as follows

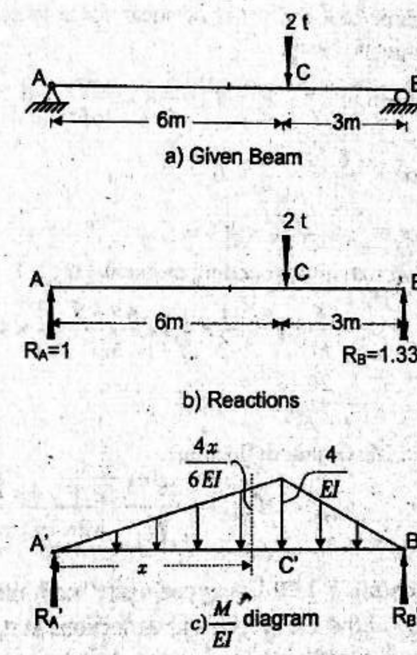


Fig. 2.36

$R_A = \dfrac{2\times3}{9} = 0.667$

$\left(\because R_A = \dfrac{Wb}{\ell}\ and\ b=3m\right)$

$R_B = \dfrac{2\times6}{9} = 1.33$

$\left(\because R_B = \dfrac{Wa}{\ell}\ and\ a=6m\right)$

$M_c = R_B\times3 = 1.33\times3 = 4\ tm$

Since the load is not symmetrical, it is necessary to take moment either about *A'* or *B'* of the conjugate beam to find the reaction forces.

Taking moment about *A*,

$$\Sigma M_A = 0$$

or, $R_B'\times9-\dfrac{1}{2}\times6\times\dfrac{4}{EI}\times\dfrac{2}{3}\times6-\dfrac{1}{2}\times3\times\dfrac{4}{EI}\left(6+\dfrac{3}{3}\right)=0$

or, $R_B' = \dfrac{1}{9EI}(48+42) = \dfrac{10}{EI}$

$$\Sigma F_y = 0$$

or, $R_A' + R_B' = \dfrac{1}{2}\times9\times\dfrac{4}{EI}$

or, $R_A' = \dfrac{18}{EI}-\dfrac{10}{EI}=\dfrac{8}{EI}$

Bending moment at $C'\ (M_c') = \dfrac{10}{EI}\times3-\dfrac{1}{2}\times3\times\dfrac{4}{EI}\times\dfrac{3}{3}$

$$= \frac{30}{EI} - \frac{6}{EI} = \frac{24}{EI} \; tm.$$

$$\therefore \text{Deflection at } C = \frac{24}{EI} = \frac{24}{2\times10^4\times10^{-1}\times10^4\times5131.6\times10^{-8}} m$$

$$= 2.34 \; cm \; \underline{\text{Ans.}}$$

aximum deflection occurs at the portion AC (as $AC > CB$) at a portion where force in the conjugate beam changes its sign (or its value $=0$). Let us me that the value of shear force is zero at a distance x form A' in the conjugate beam.

Then, $\quad F_x = R'_A - \frac{1}{2}\times x\times x\times\frac{4x}{6EI} = 0$

or, $\quad \frac{8}{EI} - \frac{x^2}{3EI} = 0$

or, $\quad x = \sqrt{24} = 4.9 \; m$

Now, maximum bending moment (M'_{max})

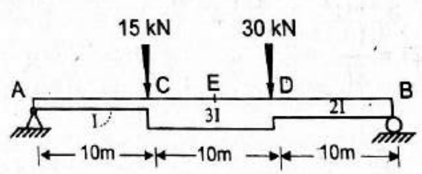
$$= \frac{8}{EI}\times4.9 - \frac{1}{2}\times4.9\times\frac{4\times4.9}{6EI}\times\frac{1}{3}\times4.9$$

$$= \frac{26.13}{EI}$$

$\therefore$ Maximum deflection

$$y_{max} = M'_{max} = \frac{26.13}{EI} = \frac{26.13}{2\times10^7\times5131.6\times10^{-8}} m = 2.55 \; cm \; \underline{\text{Ans.}}$$

Example # 2.20 Using conjugate beam method for the beam shown in Fig. (2.37) Find the slopes and deflections at A, B, C and D. Take $E = 200 \times 10^6$ kN/m^2, $I = 300 \times 10^{-4}$ m^4 and neglect the weight of the beam.



a) Given Beam

Fig. 2.37

Sol^n. The reaction force at B is determined by taking moment about A.

$R_B \times 30 - 30 \times 20 - 15 \times 10 = 0$

or, $R_B = 25 \; kN$

$\Sigma F_y = 0$

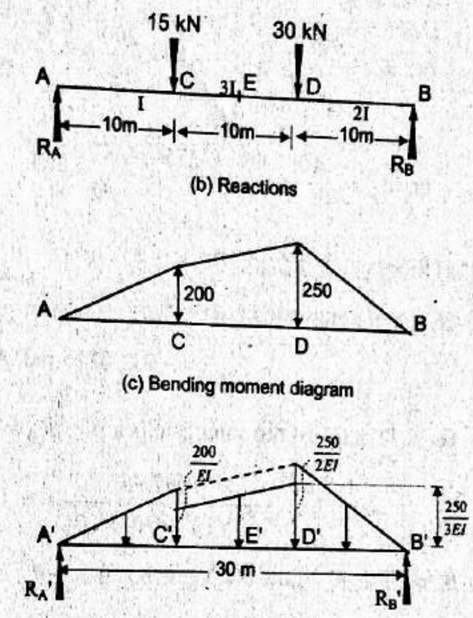
or, $R_A + R_B - 15 - 30 = 0$

or, $R_A = 20 \; kN$.

$B.M.D = R_B \times 10 = 250,$

$B.M.$ at $C = R_A \times 10 = 200 \; kN$

The $B.M.$ diagram is shown in Fig. (b). The conjugate beam loaded with M/EI diagram is shown in Fig. (d). Note that the load for conjugate beam here does not take the exact shape of bending moment diagram as in the other problems with constant I. The change in loading (M/EI diagram) due to variation of I is clear from the Fig. (c).



(b) Reactions

(c) Bending moment diagram

(d) Conjugate Beam loaded with M/EI diagram

Fig. 2.38

Now to find reaction R'_B, take moment about A such that

$$R_B'\times30 - \frac{1}{2}\times10\times\frac{250}{2EI}\times\left(20+\frac{10}{3}\right) - \left[\frac{\left(\frac{200}{3EI}+\frac{250}{3EI}\right)}{2}\times10\times\right.$$

$$\left.\left[10+\left\{\frac{\left(\frac{200}{3EI}+\frac{2\times250}{3EI}\right)}{\left(\frac{200}{3EI}+\frac{250}{3EI}\right)}\right\}\times\frac{10}{3}\right]\right] - \frac{1}{2}\times10\times\frac{200}{EI}\left(\frac{2}{3}\times10\right) = 0$$

or, $\quad R'_B \times 30 = \frac{14583.33}{EI} + \frac{11388.75}{EI} + \frac{6666.67}{EI}$

or, $\quad R'_B = \frac{1087.96}{EI}$

Using the condition

$\Sigma F_y = 0$

or, $R'_A + R'_B -$ Total downward loading $= 0$

$$\text{or, } R_A' + \frac{1087.96}{EI} - \left[\frac{1}{2}\times 10\times\frac{200}{EI} + \frac{\left(\frac{200}{3EI}+\frac{250}{3EI}\right)}{2}\times 10 + \frac{1}{2}\times 10\times\frac{250}{2EI}\right] = 0$$

$$\text{or, } R_A' = -\frac{1087.96}{EI} + \frac{2375}{EI} = \frac{1287.04}{EI}$$

a) <u>Slopes at A, B, C and D</u>

$$SF \text{ at } A \text{ of conjugate beam} = \theta_A = -R_A' = \frac{1287.04}{EI} = \frac{1287.04}{200\times 10^6\times 300\times 10^{-4}}$$

$$= -0.000215 \text{ rad. } \underline{\text{Ans.}}$$

$$= SF \text{ at } B' \text{ of the conjugate beam } = R_B' = \frac{1087.96}{EI}$$

$$= \frac{1087.96}{200\times 10^6\times 300\times 10^{-4}} = 0.000181 \text{ rad. } \underline{\text{Ans.}}$$

$$\theta_c = SF \text{ at } C' \text{ of the conjugate beam } = -R'_A + \frac{1}{2}\times 10\times\frac{200}{EI}$$

$$= -\frac{1287.04}{EI} + \frac{1000}{EI} = \frac{287.04}{EI}$$

$$= \frac{287.04}{200\times 10^6\times 300\times 10^{-4}} = 0.00005 \text{ rad.}$$

$$\theta_D = SF \text{ at } D' \text{ of the conjugate beam } = R_B' - \frac{1}{2}\times 10\times\frac{250}{2EI}$$

$$= \frac{1087.96}{EI} - \frac{625}{EI}$$

$$= \frac{462.96}{EI} = \frac{462.96}{200\times 10^6\times 300\times 10^{-4}} = 0.000077 \text{ rad. } \underline{\text{Ans.}}$$

b) <u>Deflection at A, B, C and D</u>

Deflection at A and $B = 0$

Deflection at $C = BM$ of conjugate beam at C'

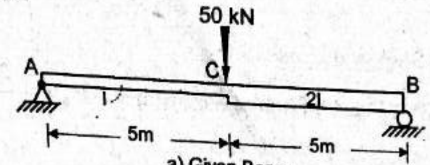
$$= R_A'\times 10 - \frac{1}{2}\times 10\times\frac{200}{EI}\times\frac{10}{3}$$

$$= \frac{1287.04}{EI}\times 10 - \frac{20000}{6EI}$$

$$= \frac{9537.07}{EI} = \frac{9537.07}{200\times 10^6\times 300\times 10^{-4}}$$

$$= 0.00159\ m = 1.59\ mm\ \underline{\text{Ans.}}$$

Example # 2.21 Determine the maximum deflection and slope at the ends in the beam shown in the Fig. (2.39). Take $E = 200\times 10^6\ kN/m^2$ and $I = 100\times 10^{-6}\ m^4$ neglect the weight of the beam.



a) Given Beam

Fig. 2.39

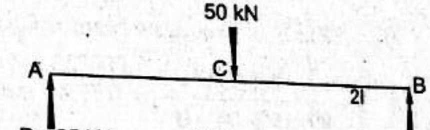
Sol^n.

The real beam is shown in Fig. (a)

Due to the symmetrical loading

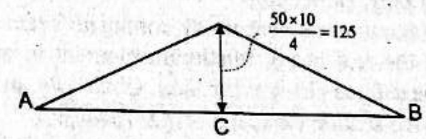
$$R_A = R_B = \frac{50}{2} = 25$$

B.M. diagram and the conjugate beam loaded with *M/EI* diagram is shown is Fig. (b) and (c) respectively.

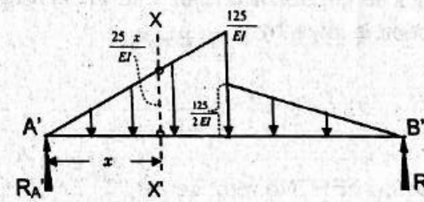


(b) Reactions

Note that the load for conjugate beam here does not take the exact shape of bending moment diagram as in the other problems with constant I. The change in loading (M/EI diagram) due to variation of I is clear from the Fig. (c).



(c) Bending moment diagram



(d) Conjugate Beam loaded with M/EI diagram

Fig. 2.40

Now, find the reactions R_A' and R_B' of conjugate beam.

Take moment about A,

$$R_B'\times 10 - \frac{1}{2}\times 5\times\frac{125}{EI}\times\frac{2}{3}\times 5 - \frac{1}{2}\times 5\times\frac{125}{2EI}\times\left(5+\frac{5}{3}\right) = 0$$

$$\text{or,}\quad R_B'\times 10 = \frac{1041.67}{EI} + \frac{1041.67}{EI} = \frac{2083.33}{EI}$$

$$\therefore R_B' = \frac{208.33}{EI}$$

$$\sum Fy = 0$$

$$R_A' + R_B' - \frac{1}{2}\times 5\times\frac{125}{EI} - \frac{1}{2}\times 5\times\frac{125}{2EI} = 0$$

$$\text{or,}\quad R_A' = -\frac{208.33}{EI} + \frac{312.5}{EI} + \frac{156.25}{EI} = \frac{260.42}{EI}$$

a) Slopes at ends

$$\theta_A = SF \text{ at } A' \text{ of conjugate beam } = -R_A' = -\frac{260.42}{EI}$$

$$= -\frac{260.42}{200\times10^6\times100\times10^{-6}} = -0.013\ rad.\ \underline{\text{Ans.}}$$

$$\theta_B = SF \text{ at } B' \text{ of Conjugate beam } = R_B' = \frac{208.33}{EI}$$

$$= \frac{208.333}{200\times10^6\times100\times10^{-6}} = 0.01042\ rad.\ \underline{\text{Ans.}}$$

b) Max. Deflection

Maximum moment in the conjugate beam refers to the maximum deflection in the real beam. Maximum moment in any beam occurs at a point where shear force changes the sign. Obviously, maximum deflection will not occurs in the section *CB* as it is stiffer (bigger *I*)

Let x be the distance from A' in the conjugate beam, then shear force at the section in given by

$$F_x = -R_A' + \frac{1}{2}\times x.\frac{25x}{EI}$$

Equating *SF* (F_x) to zero, we get,

$$\frac{25x^2}{2EI} = \frac{260.42}{EI}$$

$$\text{or,}\quad x = \sqrt{\frac{260.42\times2}{25}} = 4.56\ m$$

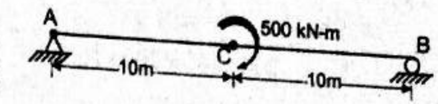
$$B.M. \text{ at } XX' = R_A' \times x - \frac{1}{2}\times4.56\times\frac{25\times4.56}{EI}\left(\frac{2}{3}\times4.56\right)$$

$$= \frac{260.42}{EI}\times4.56 - \frac{1}{2}\times4.56\times\frac{25}{EI}\times4.56\times\left[\frac{2}{3}\times4.56\right]$$

$$= \frac{397.33}{200\times10^6\times100\times10^{-6}}\ m$$

$$= 0.019867\ m\ \underline{\text{Ans.}}$$

Example # 2.22 Using conjugate beam method, determine the slope at the ends and deflection at the centre of the beam shown in Fig. (2.41). Take $E = 200\times10^6\ kN/m^2$ and $I = 200\times10^{-5}\ m^4$ neglect the weight of the beam



a) Given Beam

Fig. 2.41

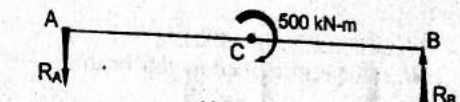
Sol^n. The reactions of the real beam is shown in the Fig. (a) where

$$\Sigma F_x = 0,$$

or $H_A = 0,$

$$\Sigma F_y = 0,$$

or $-R_A + R_B = 0$, or $R_A = R_B$



b) Reactions

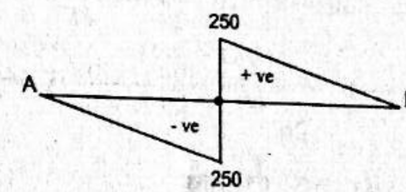
$$\Sigma M_A = 0,$$

or $R_B\times20 = 500$

$$R_B = \frac{500}{20} = 25$$

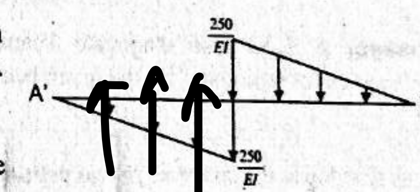
$$\therefore R_A = R_B = 25$$

$$M_c = R_B\times10 = 250$$



c) Bending moment diagram

The conjugate beam loaded with *M/EI* diagram is shown in Fig. (c)



d) Conjugate beam loaded with $\frac{M}{EI}$ diagram

Fig. 2.42

Finding reactions of the conjugate beam,

$\Sigma M_A' = 0$

$$R_B'\times20 - \frac{1}{2}\times10\times\frac{250}{EI}\times\left(10+\frac{10}{3}\right) + \frac{1}{2}\times10\times\frac{250}{EI}\left(\frac{2}{3}\times10\right) = 0$$

$$\text{or,}\quad R_B'\times20 = -\frac{16666.67}{EI} + \frac{8333.33}{EI}$$

$$\text{or,}\quad R_B' = -\frac{416.67}{EI}$$

$$R_A' = +\frac{416.67}{EI}$$

a) <u>Slope at ends</u>

Slope at the end A of real beam (θ_A) = Shear at end A' of conjugate beam $= R_A'$

$$\theta_A = \frac{416.67}{EI} = \frac{416.6}{200\times10^6\times200\times10^{-5}} = +0.00104\ rad.\ \underline{\text{Ans.}}$$

Slope at the end B of real beam (θ_B) = Shear at end B' conjugate beam $= R_B'$

or,

$$\theta_B = \frac{416.67}{EI} = 0.00104\ \underline{\text{Ans.}}$$

b) <u>Deflection at the centre y_c</u>

B.M. at the centre of conjugate beam,

$$M_c' = R_A' \times 10 - \frac{1}{2}\times10\times\frac{250}{EI}\times\frac{10}{3}$$

$$= \frac{416.67}{EI}\times10 - \frac{1}{2}\times10\times\frac{250}{EI}\times\frac{10}{3}$$

$$= 0$$

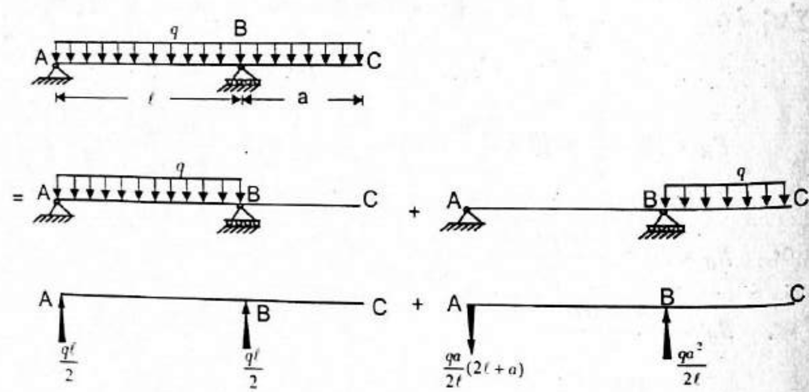
$\therefore\ y_c = M_c' = 0$ <u>Ans.</u>

Hence deflection at C is zero

Example # 2.23 Use conjugate beam method to determine vertical displacement of free end C for the given beam. Take EI constant for beam.

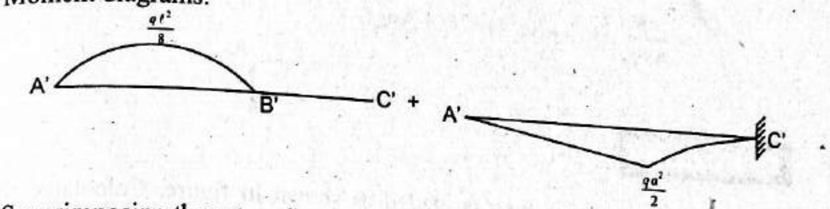
[2058 Shrawan]

Let us first break the structure in two parts as given below



Reactions

Moment diagrams:



Superimposing these two figures, we get conjugate beam diagram,

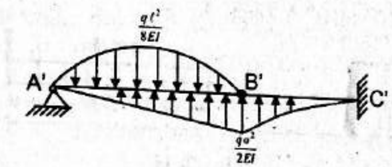


Fig. 2.43

Reaction R_1 at B' due to *udl* at $A'B'$

$$= \frac{1}{2}\times\frac{2}{3}\times\frac{q\ell^2}{8EI}\times\ell = \frac{q\ell^3}{24EI}$$

Again reaction R_2 at B' due to triangular loading on $A'B'$

$$= R_2\times\ell = \frac{1}{2}\times\ell\times\frac{qa^2}{2EI}\times\frac{2}{3}\ell$$

$$\therefore\quad R_2 = \frac{qa^2\ell}{6EI}$$

$\therefore$ Net reaction on the hinge B' due to loads on $A'B'$

$$R_B' = R_1 - R_2 = \frac{q\ell^3}{24EI} - \frac{qa^2\ell}{6EI}$$

$$= \frac{q\ell}{6EI}\left[\frac{\ell^2}{4} - a^2\right]$$

$$= \frac{q\ell}{24EI}(\ell^2 - 4a^2)$$

Now, *B.M.* at C' M'_C= Deflection at C,

$$y_c = \frac{1}{3}\times a\times\frac{qa^2}{2EI}\times\frac{3a}{4} - R_{B'}\times a$$

$$= \frac{qa^4}{16EI} - \frac{q\ell}{24EI}\left(\ell^2 - 4a^2\right)a$$

$$= \frac{qa}{8\,EI}\left(\frac{a^3}{2} - \frac{\ell}{3}\left(\ell^2 - 4a^2\right)\right)$$

$$= \frac{qa}{48\,EI}\left(3a^3 + 8a^2\ell - 2\ell^3\right) \text{ Ans.}$$

2.9 EXERCISES

Ex .1 A cantilever $3m$ long is loaded as shown in figure. Calculated the deflection at the free and if the section is rectangular, (200×350) mm. Take $E = 0.105 \times 10^5\ N/mm^2$

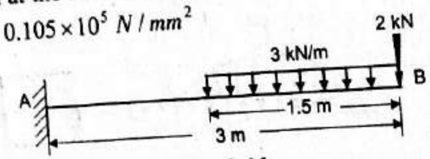


Fig. 2.44

$(2.399 + 3.457) = 5.86\ mm$ Ans.

Ex.2 A uniform beam of length L is simply and symmetrically supported as shown below. Find the ration $1/\ell$ so that the upward deflection at each end equals the downward deflection at mid span, due to concentrated load.

(Ans: 5/3)

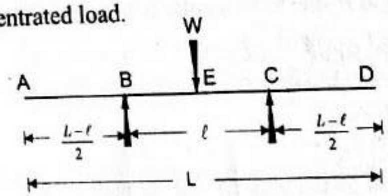


Fig. 2.45

Ex. 3 A beam AB of 4 meters span is simply supported at the ends and is loaded as shown in figure. Determine by using Macaulay's Method
a) Deflection at C b) Max. deflection c) Slope at the end A.
Take $E = 200 \times 10^6\ kN/m^2$ and $I = 20 \times 10^{-6}\ m^4$

(Ans: 8.74mm downward, 8.75mm, -0.417^0)

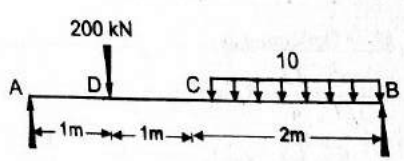


Fig. 2.46

Ex. 4 A beam 6 m long is loaded as shown in figure. If the flexural rigidity (EI) of beam is $8 \times 10^4\ kN/m^2$, find the deflection at point C. Use Maccaulay's Method.

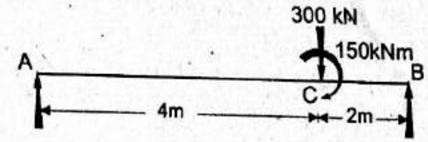


Fig. 2.47

Ex. 5 A beam of 4 m span carrying a point load of 40 kN at a distance of 3 m from the left end. Calculate the slope at the two supports and deflection under the load. Also calculate maximum deflection. Take $EI = 2.6 \times 10^7\ N-m^2$ (Use moment area method)

(Ans: θ_A = 0.0009615 radians, θ_B = 0.001346 radians.
Deflection = 1.0154 mm, max. deflection = 1.37 mm)

Ex. 6 A rectangular beams 150 $mm \times 240\ mm$ deep is simply supported at the ends on a span of 4 m and carries a uniformly distributed load of 6 kN/m on whole span. What point load at the centre should it carry so that maximum deflection is doubled.

(Ans: 30 kN)

Ex. 7 Using Conjugate beam method, calculate the slope at C and A and deflection of the central point D of the beam of uniform cross-section shown in figure. Take $E = 200 \times 10^6\ kN/m^2$, $I = 120 \times 10^{-4}\ m^2$

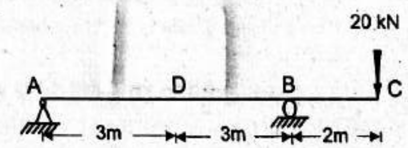


Fig. 2.48

(Ans: θ_c = 0.005 radian, θ_A = 0.00166 radian and y_D = 3.7 mm)

Ex. 8 Using conjugate beam method, calculate the UDL W over a beam shown in the figure so that deflection at the free end does not exceed 100 mm. Take $E = 200 \times 10^6\ kN/m^2$ and $I = 2.72 \times 10^{-4}\ m^2$

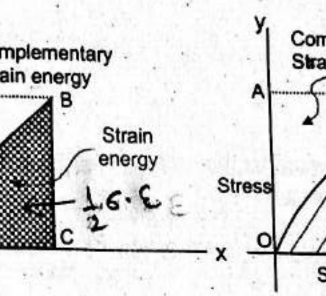
(Ans: 40.26 kN/m)

3

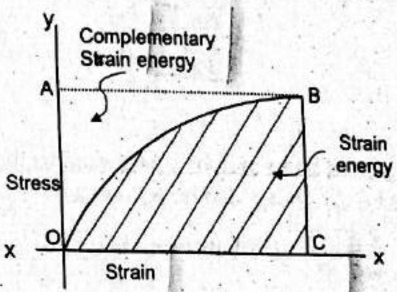
STRAIN ENERGY

3.1 STRAIN ENERGY AND COMPLEMENTARY STRAIN ENERGY

If we load an elastic material, it deforms. The deformation increases as the load is gradually increased. We can plot a graph between the load and the corresponding deformations, which will be a straight line for linear materials and curve for non-linear materials as indicated in the figure given below.



a) Load- deformation curve for linear elastic material



b) Load- deformation curve for non- linear elastic material

Fig. 3.1

The area *OBC* that is under the line *OB* represents the strain energy stored in the material due to the load where as the area *OAB* above *OB* is known as complementary strain energy. For linearly elastic materials, these areas (*OBC* and *OAB*) are equal.

3.2 STRAIN ENERGY DUE TO AXIAL STRESS

Consider the infinitesimal element acted upon by normal stress σ_x as shown in figure.

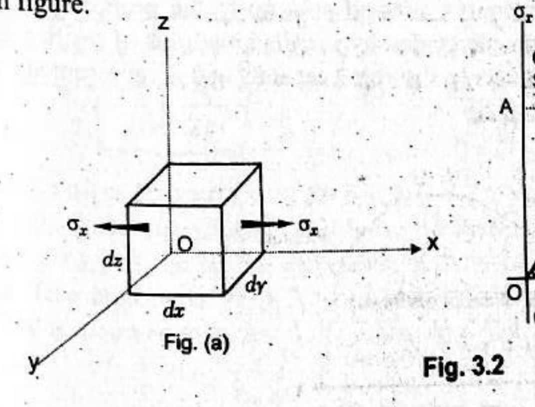


Fig. (a)

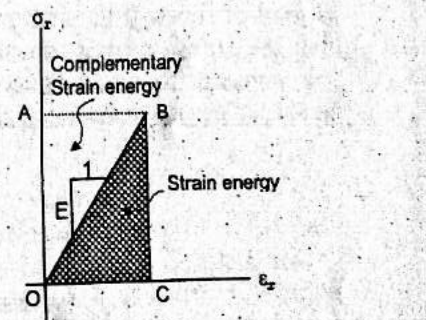


Fig. (b)

Fig. 3.2

The force that acts on the face of the element would be $\sigma_x.dy.dz$ where $dy.dz$ is the area of the face. Due to this force, the element elongates by $\varepsilon_x.dx$ where ε_x is the strain. As this force is applied gradually from zero to its final value, the average force during loading will be $1/2\,\sigma_x.dy.dz$. Now strain energy stored is the work done by this force, which can be expressed as

$$dU = \underbrace{\frac{1}{2}.\sigma_x.dy.dz}_{\text{Average force}} \times \underbrace{\varepsilon_x.dx}_{\text{distance}} = \frac{1}{2}.\sigma_x.\varepsilon_x.dx.dy.dz = \frac{1}{2}.\sigma_x.\varepsilon_x.dv \quad \text{...........(3.1)}$$

Where dv is the volume of the element

The strain energy stored in an elastic body per unit volume is called strain energy density (U_0) and it can be obtained by the above equation.

$$U_0 = \frac{dU}{dV} = \boxed{\frac{\sigma_x.\varepsilon_x}{2}} = \frac{\sigma_x^2}{2E} \quad \text{.......... (3.2)}$$

$$(\because \varepsilon_x = \frac{\sigma_x}{E})$$

This expression may be interpreted as the area of triangle OAB in Fig. (3.2-b).

The Eq. (3.1) can also be written as

$U = \int_{vol} \frac{\sigma_x \varepsilon_x}{2}.dv$, Substituting $\varepsilon_x = \frac{\sigma_x}{E}$ as given by Hook's law, we obtain the equation for strain energy stored.

$$U = \int_{vol} \frac{\sigma_x^2}{2E}.dV \quad \text{.......... (3.3)}$$

Substituting $\int_{dv} dV = A.\ell$ and $\sigma_x = \frac{P}{A}$ for axial stress

we get, $U = \frac{P^2}{A^2}.\frac{1}{2E}.A.\ell$ $\qquad \therefore \boxed{U = \frac{P^2\ell}{2AE}}$

Eq. (3.2) shows that the strain energy stored in a material is proportional to its level of stress. If the material is stressed only up to the proportionality limit, the corresponding strain energy density is called modulus of resilience. This represents the material's energy absorbing capacity and is represented by the shaded portion in the Fig. (3.3).

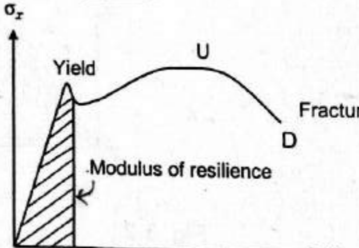


Fig. 3.3

Another term called toughness refers to the ability of the material to absorb energy without fracturing. This is given by the area under a complete stress-strain diagram. Larger the total area under the stress strain diagram, the tougher the material will be. If the modulus of resilience is higher, we can say that the material is stronger but not necessarily tougher. This is illustrated in the diagram given below.

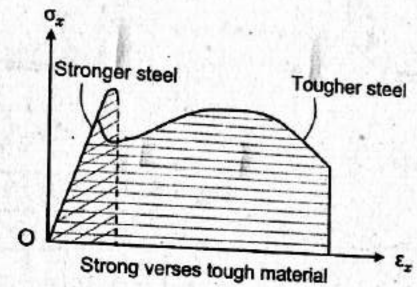


Fig. 3.4

3.3 Strain Energy due to Bending

Although, bending stress varies triangularly by its nature, it still acts normal to any cross section. Therefore Eq. (3.3) can be used by substituting $\sigma = (M/I).y$ to calculate strain energy.

$$U = \int_v \frac{\sigma_x^2 dv}{2E}$$

$$= \int_v \frac{M^2}{I^2}.y^2.\frac{1}{2E}.dv$$

$$= \int_a \frac{M^2}{I^2}.y^2.\frac{1}{2E}.da.dx \qquad (\because dv = da.dx)$$

$$= \int_\ell \left[\frac{M^2}{2EI^2}.\left\{\int_a y^2 da\right\}\right].dx$$

But $\int y^2 da = I$

$$\therefore \boxed{U = \int_\ell \frac{M^2}{2EI}.dx}$$

3.4 Strain Energy due to Shear

Let the infinitesimal element is under the system of shear stress (τ) as shown in fig. (3.5-a). Under the application of the shear force, the topside of one of the face considered (Fig. 3.5-b) moves by a distance $r.dy$ while the bottom fibre is assumed to be fixed.

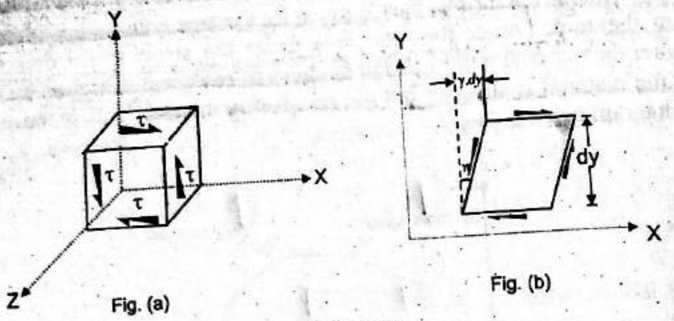


Fig. 3.5

As before,

$$dU=\underbrace{\frac{1}{2}\,\tau.(dx.dz)}_{\text{Average force}}\times\underbrace{\gamma.dy}_{\text{distance}}=\frac{1}{2}\tau\gamma(dx.dy.dz)=\frac{1}{2}.\tau\gamma.dv \quad\text{.............(3.4)}$$

As $\tau=G.\gamma$, $\gamma=\frac{\tau}{G}$ Substituting this is in Eq. (3.4)

$$\boxed{U_{shear}=\int_v\frac{\tau^2}{2G}dv}\quad\text{.............................(3.5)}$$

We know $\tau=\frac{FQ}{It}$, and substituting this in Eq. (3.5)

$$U=\int_v\frac{1}{2G}\left\{\frac{VQ}{It}\right\}^2.dv \quad\text{.............................(3.6)}$$

For rectangular section, as given by the Fig. (3.6) shown below.

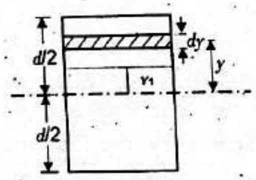


Fig. 3.6

$$Q=\int_{y_1}^{d/2}b.y.dy$$

or, $$Q=\frac{b}{2}\left(\frac{d^2}{4}-y_1^2\right)$$

Substituting the value of Q in Eq. (3.6) and re-writing $dv=dx.dy.dz$,

$$U=\int_\ell\frac{F^2}{2GI^2}\left\{\int_A\left(\frac{Q}{t}\right)^2dy.dz\right\}dx \quad\text{.............................(3.7)}$$

But since, $\int_A\left(\frac{Q}{t}\right)^2dy.dz=\int_{-d/2}^{d/2}\frac{b}{4}\left(\frac{d^2}{4}-y^2\right)^2dy=\frac{bd^5}{120}$, $(t=b)$

$$I=\frac{bd^3}{12}\text{ and }A=bd$$

$$U=\int_\ell\frac{1.2V^2}{2GA}.dx$$ Shear force $F=V$

$$\therefore\quad\boxed{U=K\int_0^\ell\frac{V^2}{2GA}.dx}$$

Where K is a constant. Its value depends on the shape of the section. For rectangular section, $K=1.2$

3.5 Strain Energy due to Torsion

Strain energy due to torsion can be obtained by replacing τ in Eq. (3.5)

$$U=\int\frac{\tau^2}{2G}.dv$$

Substituting $\tau=\frac{T\rho}{J}$ and, where T= torque, J= polar moment of inertia.

$$U=\int_\ell\frac{T^2}{2GJ^2}\left\{\int_A\rho^2.dy.dz\right\}.dx$$

But, $\int_A\rho^2.dy.dz=J$

$$\therefore\quad\boxed{U=\int\frac{T^2.dx}{2GJ}}$$

Thus, for a prismatic bar of length ℓ subjected to axial force, moment, shear and torsional forces, the total strain energy stored would be,

$$U=\int_0^\ell\frac{P^2dx}{2AE}+\int_0^\ell\frac{M^2dx}{2EI}+K\int_0^\ell\frac{V^2dx}{2GA}+\int_0^\ell\frac{T^2dx}{2GJ}\quad\text{.............(3.8)}$$

3.6 Work and Complementary Work

We know that a structure deforms when a force is applied on it. The product of the force and the deformation gives the work done by the force. The work done is represented graphically by the area under the force deformation curve line shown in Fig (3.7)

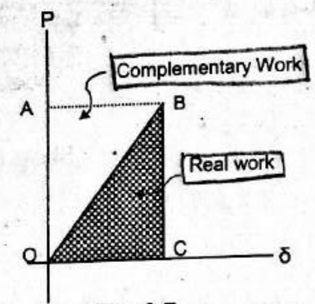


Fig. 3.7

The area above the line OB is complementary work such that for a linear system, the sum of work and the complementary work is equal to zero.

3.7 Deflection by Method of Real Work

As described earlier, the external work done (W) which is the input energy in the structure, is stored as the internal strain energy (U). By the principle of conservations of energy, these two quantities (W and U) can be equated to find the deflection of the structure. The procedure is illustrated in the following examples. This method of finding the deflection is also called as "Real work method" since work done by the actual loads are considered. This method however faces problem when there are several loads applied in the structure and deflection is desired not at the point of application of the loads.

3.8 Deflection by Strain Energy Method

By using principle of conservation of energy $W_{ext} = W_{int}$ one can also find the deflection of a structure. External work done is the product of force and defection and internal work is the strain energy of the structures due to the applied force.

$$W_{ext} = \text{Force} \times \text{deformation/deflection.} \quad \text{..............................} \quad (3.9)$$

$$W_{int} = \int_0^{\ell} \frac{\sigma^2}{2E}.dx \quad \text{..} \quad (3.10)$$

Equating (3.9) and (3.10) the unknown deflection is obtained.

Example # 3.1 Calculate the elongation, stress developed and strain energy stored in a steel bar 5 m in length and 1200 mm^2 in cross section when a tensile load of 110 t is gradually applied. Take $E = 2\times10^5\ N/mm^2$.

Solⁿ.

We have,

$$U = \frac{\sigma^2 V}{2E} = \frac{P^2 \ell}{2AE} = \frac{(110\times1000\times10)^2 \times 5\times1000}{2\times1200\times2\times10^5}$$

$$= 12604167\ N\text{-}mm = 12.6\ kN\text{-}m\ \underline{\text{Ans.}}$$

$$\sigma = \sqrt{\frac{2EU}{V}} = \sqrt{\frac{2EU}{A\times\ell}} = \sqrt{\frac{2\times2\times10^5\times12604167}{1200\times5\times1000}}$$

$$= 916.66\ N/mm^2\ \underline{\text{Ans.}}$$

Example # 3.2 A square steel bar 40 mm × 40 mm in section, 3 m long is subjected to an axial pull of 128 kN. Taking E = 200 GN/m^2, calculate the elongation of the bar. Also, calculate the energy stored in the bar during the extension. Take sectional area of the bar = 40 mm × 40 mm = 16 × 10^{-4} m^2.

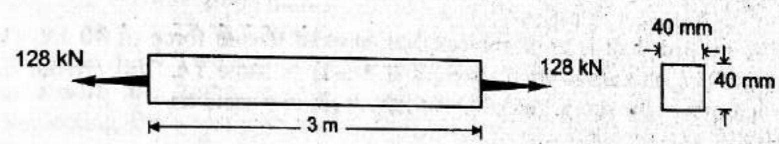


Fig. 3.8

Span $\ell = 3\ m$

$$E = 200\ GN/m^2$$

Elongation of the bar $\delta\ell = \frac{P\ell}{AE} = \frac{(128\times1000)\times3}{16\times10^{-6}\times200\times10^9} = 1.2\ mm$. $\underline{\text{Ans.}}$

$$\sigma = \frac{128\times1000}{16\times10^{-4}} = 8\times10^7\ N/m^2$$

Energy stored in the bar during elongation U is

$$U = \frac{\sigma^2}{2E}.A\ell,$$

$$= \frac{(8\times10^7)^2\times16\times10^{-4}\times3}{2\times2\times10^{11}}$$

$$= 76.8\ Nm \quad \text{or, } joules\ \underline{\text{Ans.}}$$

Example # 3.3 The steel and brass bars comprising the stepped shaft are to suffer the same displacement under tensile force of 40 kN. Determine the diameter of brass bars d_b, the stress in each materials and total strain energy stored. Take, $E_{steel} = 207\ Gpa$ and $E_{brass} = 82.7\ Gpa$.

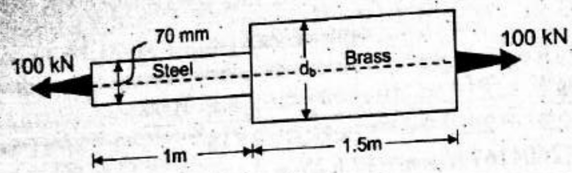


Fig. 3.9

Sol^n.
Given for steel
Diameter $d_s = 70\ mm = 0.7\ m$.
Length $\ell = 1m$
For brass $E_s = 207\ Gpa$.
Tensile force $(P) = 100\ kN$.
Let the stresses in the materials be σ_s, σ_b for steel and brass and δ_s and δ_b be their displacements.

The stepped shaft is to be subjected to an axial tensile force of 40 kN. The total force on each section, (steel and brass) is same i.e. 100 kN but the intensities of the stresses should be different for two sections.
Hence,

$$\sigma_s = \frac{P}{A_s} = \frac{100\times10^{-3}}{\frac{\pi}{4}\times(0.07)^2} = 25.98\ MN/m^2$$

Also, $\delta_s = \delta_b$ (Given)

$$\text{or, } \frac{P\ell_s}{A_sE_s} = \frac{P\ell_b}{A_bE_b}$$

$$\frac{\sigma_s\ell_s}{E_s} = \frac{\sigma_b\ell_b}{E_b} \qquad (\text{As } \frac{P}{A} = \sigma)$$

$$\therefore \quad \sigma_b = \frac{\sigma_s\ell_s}{E_s}\times\frac{E_b}{\ell_b}$$

$$= \sigma_s\left(\frac{\ell_s}{\ell_b}\right)\left(\frac{E_b}{E_s}\right)$$

$$= 25.984\times\left(\frac{1}{1.5}\right)\times\left(\frac{82.7}{207}\right)$$

or, $\sigma_b = 6.92\ MN/m^2$ Ans.

Diameter of brass bar, d_b

$$P = \sigma_b \times A_b$$

$$100\times10^{-3} = 6.92\times\frac{\pi}{4}\times d_b^2$$

$$\therefore \quad d_b = \left[\frac{100\times10^{-3}\times4}{6.92\times\pi}\right]^{\frac{1}{2}}$$

$$= 0.13564\ m$$

$$= 135.64\ mm$$

Total Strain energy stored

$$U = U_s + U_b$$

$$= \frac{\sigma_s^2A_s\ell_s}{2E_s} + \frac{\sigma_b^2A_b\ell_b}{2E_b}$$

$$= \frac{(25.98)^2\times\frac{\pi}{4}\times(0.07)^2\times\ell}{2\times(207\times10^3)} + \frac{(6.92)^2\times\frac{\pi}{4}\times(0.13564)^2\times15}{2\times82.7\times10^3}$$

$$= 1.257\times10^5\ MN/m \quad = 12.57 N-M$$

$$= 12.56 \text{ Joule Ans.}$$

Example # 3.4 Two elastic bars, whose proportions are shown in figure, are to absorb the same amount of energy delivered by the axial forces. Neglecting the stress concentrations, compare the stresses in the two bars. The cross -sectional area of the left bar is A and that of the right bar is A and $2A$ as shown in figure.

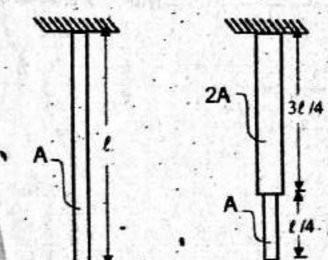


Fig. 3.10

Sol^n. $U_1 = \frac{\sigma_1^2 v}{2E} = \frac{\sigma_1^2 A\ell}{2E}$

$$U_2 = \frac{\sigma_2^2}{2E}\int_{lower\ part} dv + \frac{\left(\frac{\sigma_2}{2}\right)^2}{2E}\int_{upper\ part} dv$$

(Stress in the upper part is half as the area is double of the lower part)

$$= \frac{\sigma_2^2}{2E}\left(\frac{A\ell}{4}\right) + \frac{\sigma_2^2}{8E}.2A\ell\times\frac{3}{4}$$

$$= \frac{\sigma_2^2}{2E}\left(\frac{A\ell}{4}+\frac{6A\ell}{16}\right) = \frac{\sigma_2^2}{2E}\left(\frac{4A\ell+6A\ell}{16}\right) = \frac{\sigma_2^2}{2E}.\frac{5}{8}A\ell$$

According to the question,

$$\sigma_1^2 \frac{A\ell}{2E} = \frac{\sigma_2^2}{2E} \cdot \frac{5}{8} A\ell$$

$$\therefore \quad \sigma_1 = 0.79\sigma_2 \quad \underline{\text{Ans.}}$$

Example # 3.5 Two bars of same material and same length are subjected to equal and gradually applied tensile load. One bar is $2d$ in diameter throughout and the other has the diameter d over the middle third at its length, the remainder having a diameter $2d$. Compare the strain energy of the two bars.

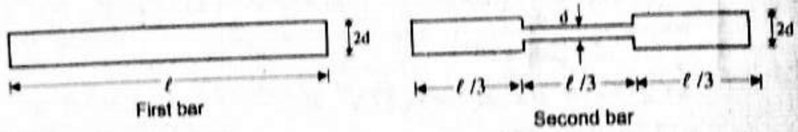


Fig. 3.11

Sol^n. Strain energy of the first bar $U_1 = \frac{\sigma^2 V}{2E}$, $\sigma = \frac{P}{A}$

or, $U_1 = \frac{P^2}{A^2} \cdot \frac{A.\ell.}{2E} = \frac{P^2\ell}{2AE}$ $\quad \left(\text{But } \because A = \frac{\pi d^2}{4}\right)$

$$= \frac{P^2\ell}{2 \times \frac{\pi \times (2d)^2}{4} \times E}$$

$$= \frac{P^2\ell}{2\pi d^2 E}$$

Strain energy of the 2nd bar = Strain energy of two end portions + Strain energy of middle portion.

or, $U_2 = 2\frac{P^2\ell_1}{2A_1E} + \frac{P^2\ell_2}{2A_2E}$

$$= P^2\left(\frac{\ell}{3}\right)\frac{1}{2\times\pi\frac{(2d)^2}{4}} \times \frac{1}{E} + \frac{P^2\ell}{2\times\frac{\pi d^2}{4}} \times \frac{1}{E}$$

$$= \frac{P^2\ell}{\pi d^2 E}$$

$$U_1 : U_2 = \frac{\frac{P^2\ell}{2\pi d^2 E}}{\frac{P^2\ell}{\pi d^2 E}} = 1:2 \quad \underline{\text{Ans.}}$$

(Alternatively, strain energy of the second bar may be found out by considering two small cut outs at the middle of the beam from the first bar.)

Example # 3.6 Find the elastic strain energy in a rectangular cantilever beam subjected to a bending moment M applied at the end.

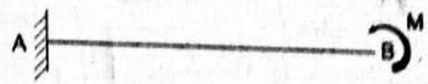


Fig. 3.12

Sol^n. Due to the application of moment M at point B, bending moment remains constant along the length of the beam and its value is M itself.

We have, $U = \int_0^\ell \frac{M^2}{2EI}dx = \frac{M^2}{2EI}\int_0^\ell dx = \frac{M^2\ell}{2EI}$

But, $\frac{M}{I} = \frac{\sigma}{y}$, $\therefore M = \sigma.\frac{I}{y} = \sigma_{max}.\frac{bd^3}{12} \times \frac{1}{d/2} = \frac{bd^2}{6}\sigma_{max}$

or, $U = \left(\frac{bd^2\sigma_{max}}{6}\right)^2 \frac{\ell}{2E} \times \frac{1}{bd^3/12} = \frac{bd\ell\sigma_{max}^2}{6E}$

$$= \frac{\sigma_{max}^2}{2E}\left(\frac{V}{3}\right) \quad \underline{\text{Ans.}}$$

Where V is the volume of the beam.

The expression shows that for a given maximum stress, the volume of material in the beam is only a third as effective for absorbing energy as would be in a uniformly stressed bar where $U = \sigma^2 / 2E(V)$. It is due to the variation of bending stress across the cross section of beam. The axial stress on the other hand is uniform along the cross section. (See Section 1.2 for details of structures with uniform and non-uniform stress forms.

Example # 3.7 Two cantilever beams of same material are similar in every aspect except size. One having all of its linear dimensions just 'n' times those of the other. What is the ratio of their strain energies when they are subjected to their own distributed weights? Consider energy due to bending only.

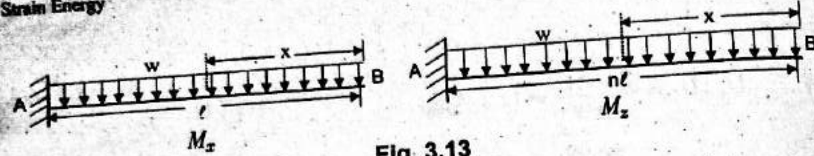


Fig. 3.13

Sol^n

$$M_x = \frac{wx^2}{2} \qquad\qquad M_x = \frac{wx^2}{2}$$

$udl(w) = \rho\, db\, d^2$ per meter

The ratio of the strain energies is given as

$$\frac{U_1}{U_2} = \frac{\int_0^{\ell} \frac{M_x^2\,dx}{2EI}}{\int_0^{n\ell} \frac{M_x^2\,dx}{2EI}} = \frac{\frac{1}{2EI}\int_0^{\ell} \frac{\rho^2 b^2 d^2}{4}.x^4\,dx}{\frac{1}{2EI}\int_0^{n\ell} \frac{\rho^2 b^2 d^2}{4}.x^4\,dx}$$

$$= \frac{\left[\frac{x^5}{5}\right]_0^{\ell}}{\left[\frac{x^5}{5}\right]_0^{n\ell}} = \frac{\frac{\ell^5}{5}}{\frac{n^5\ell^5}{5}} = \frac{1}{n^5} \quad \underline{\text{Ans.}}$$

Example # 3.8 Find the expression for deflection of the free end of an elastic rod of constant cross sectional area A and of length ℓ due to an axial force P applied at the free end.

Sol^n.

(Force applied is gradual and it increases from 0 - P such that average force $= \frac{P}{2}$)

External work done, W_{ext} = Average force × distance

$$\frac{P}{2} \times \Delta \text{ .... (3.11)}$$

Internal work done, W_{int} = Strain energy $= \frac{P^2\ell}{2AE}$ ..... (3.12)

Equating Eq. (3.11) and Eq. (3.12), we get,

$$\Delta = \frac{P\ell}{AE} \quad \underline{\text{Ans.}}$$

(Note that this is the same expression as obtained by Hook's law: $\sigma \propto \varepsilon$, $\frac{P}{A} = \frac{\Delta}{\ell}$ $\therefore \Delta = \frac{P\ell}{AE}$)

Fig. 3.14

Example # 3.9 Find the deflection at B in the cantilever by Real work method.

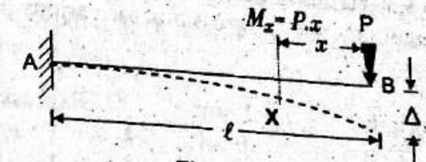


Fig. 3.15

Sol^n.

Real work done by the load P

$$W = \frac{P}{2}.\Delta \qquad (\frac{P}{2} \text{ is the average load})$$

Strain energy stored

$$U = \int_0^{\ell} \frac{M^2\,dx}{2EI}$$

$$= \int_0^{\ell} \frac{(Px)^2.dx}{2EI}$$

$$= \frac{P^2}{2EI}.\int_0^{\ell} x^2\,dx$$

$$= \frac{P^2\ell^3}{6EI}$$

Equating W and U, we get.

$$\frac{P}{2}.\Delta = \frac{P^2\ell^2}{6EI}$$

$$\therefore \quad \Delta = \frac{P\ell^2}{3EI} \quad \underline{\text{Ans.}}$$

Example # 3.10 Find the vertical deflection at point C in the frame shown below. Take $E = 200\ kN/mm^2$ and $I = 15 \times 10^6\ mm^4$.

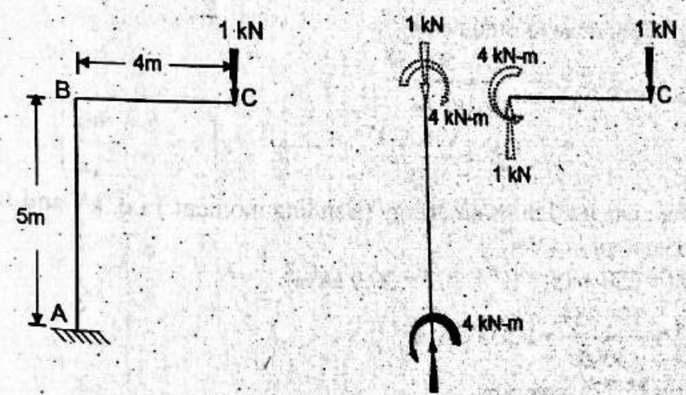


Fig. 3.16

Sol^n.
Free body diagram for various portions of the structure is shown in Fig. (3.16) and details of bending moment expressions for various portion of the structures are shown below.

Portion	origin	Limit	Expression (M)
BC	C	$0-4$	$1\times x = x$
AB	B	$0-5$	4

We have, Strain Energy $U = \int_{BC} \frac{M^2}{2EI}.dx + \int_{AC} \frac{M^2}{2EI}.dx$

$$U = \int_0^4 \frac{x^2}{2EI}.dx + \int_0^5 \frac{4^2}{2EI}.dx$$

$$= \frac{1}{2EI}\left[\frac{x^3}{3}\right]_0^4 + \frac{16}{2EI}[x]_0^5$$

$$= \frac{1}{2EI}\left[\frac{4^3}{3} - \frac{0^3}{3}\right] + \frac{16}{2EI}[5-0]$$

$$= \frac{1}{2EI}\times\frac{64}{3} + \frac{16}{2EI}\times 5$$

$$= \frac{50.67}{EI}$$

Now, External work done $W = \frac{1}{2}\times P\times\Delta$

$$= \frac{1}{2}\times 1\times\Delta$$

$$= \frac{\Delta}{2}$$

Equating Work done to Strain energy.

$$\frac{\Delta}{2} = \frac{50.667}{EI}$$

$$\therefore \quad \Delta = \frac{101.333}{EI}$$

Noting the units used in strain energy (Bending moment) i.e. kN and meters, EI should be used in $kN\text{-}m^2$.

$$EI = 200\times 15\times 10^6\times 10^{-6} = 30.0\ kN\text{-}m^2$$

$$\therefore \quad \Delta = \frac{101.333}{3000} = 0.033778\ m$$

$$= 33.778\ mm$$ **Ans.**

Example # 3.11 A beam of length ℓ is simply supported at its ends and carries a concentrated load W at its centre. It has rectangular cross-section of breadth b and depth d. If G_1 is modulus of rigidity for the beam, determine the deflection due to shear.

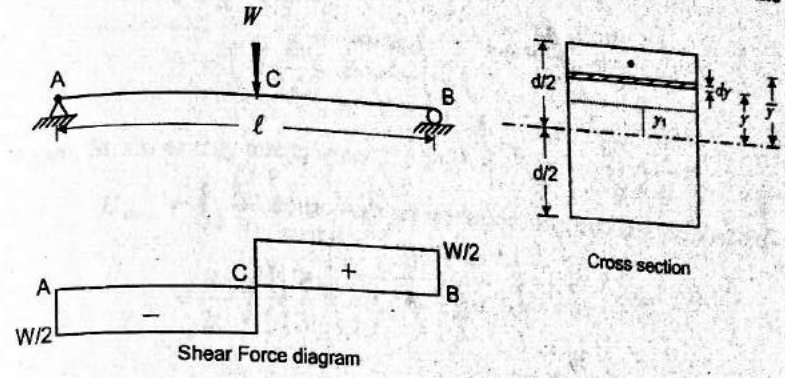


Fig. 3.17

Now, to find out strain energy in the beam
Consider a layer of thickness dy at a distance y from neutral axis,

Shear stress at the layer, $\tau = \frac{PA\bar{y}}{Ib}$

Where, P = Shear force at the section
$A\bar{y}$ = First moment area above the layer about neutral axis, and
I = Moment of inertia

$$\therefore \quad \text{Shear Stress,} \quad \tau = \frac{\frac{W}{2}b\left(\frac{d}{2}-y\right)\times\left(y+\frac{\frac{d}{2}-y}{2}\right)}{\frac{bd^3}{12}\times b}$$

$$= \frac{6W}{bd^3}\left(\frac{d}{2}-y\right)\times\left(\frac{\frac{d}{2}+y}{2}\right)$$

$$= \frac{3W}{bd^3}\left(\frac{d}{2}-y\right)\times\left(\frac{d}{2}+y\right)$$

$$= \frac{3W}{bd^3}\left[\left(\frac{d}{2}\right)^2 - y^2\right]$$

Volume of layer = $b.dx.dy$

Shear strain energy in the layer $= \dfrac{\tau^2}{2G} b.dx.dy$

$$= \frac{1}{2G}\left[\frac{3W}{bd^3}\left\{\left(\frac{d}{2}\right)^2 - y^2\right\}\right]^2 b.dx.dy$$

$$= \frac{1}{2G} b.\frac{9W^2}{b^2d^6}\left(\frac{d^2}{4} - y^2\right)^2 dx.dy$$

Total shear strain energy due to shear for the portion BC

$$(U_S)_{BC} = \frac{1}{2G}\int_0^{\frac{\ell}{2}}\int_{-\frac{d}{2}}^{\frac{d}{2}} \frac{9W^2}{bd^6}\left(\frac{d^2}{4} - y^2\right)^2 dy.dx$$

$$= \frac{3W^2\ell}{40Gbd}$$

Similarly, $(U_S)_{CA} = \dfrac{3W^2\ell}{40Gbd}$

$\therefore \quad U_S = (U_S)_{BC} + (U_S)_{CA}$

$$= \frac{3W^2\ell}{20Gbd}$$

Deflection due to shear

$$\delta_S = \frac{\partial U_S}{\partial W} = \frac{\partial}{\partial W}\left(\frac{3W^2\ell}{20Gbd}\right) = \frac{3W\ell}{10Gbd}$$

Hence, $\delta_S = \dfrac{3W\ell}{10Gbd}$ <u>Ans.</u>

Example # 3.12 Cantilever beam of breadth b and depth d is subjected to a vertical load P at the free end as shown in figure. Compute the vertical deflection Δ at the free end of the beam. Consider shear deflection and explain its significance.

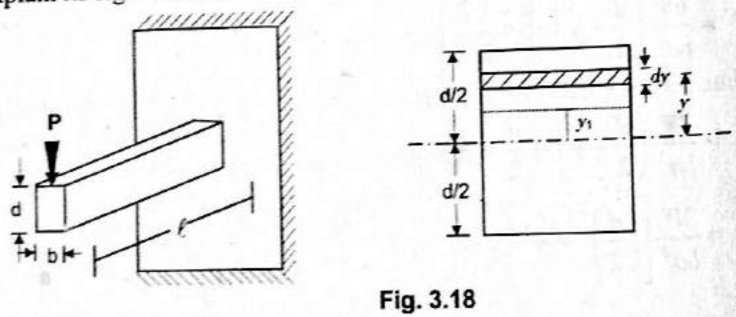


Fig. 3.18

It is observed that shear force at any section of the beam is $F = -P$ and the moment at any section is given by $M = -Px$. Now Strain Energy due to bending,

$$U_{bending} = \int_0^{\ell} \frac{M^2 dx}{2EI}$$

$$= \int_0^{\ell} \frac{(-Px)^2}{2EI}.dx = \frac{P^2\ell^3}{6EI}$$

Again, Strain energy due to shear.

$$U_{shear} = \int_{vol} \frac{\tau^2}{2G} dV = \frac{1}{2G}\int_{vol} \tau^2 dV$$

$$= \frac{1}{2G}\int_{-\frac{d}{2}}^{+\frac{d}{2}} \left\{\frac{P}{2I}\left[\left(\frac{d}{2}\right)^2 - y^2\right]\right\}^2 \ell.b.dy$$

[Alternatively, the expression $U_{shear} = k\int_0^{\ell} \dfrac{F^2 dx}{AG}$ can also be used]

$$= \frac{P^2\ell.b}{8GI^2}.\frac{d^5}{30}$$

$$= \frac{P^2.\ell.bd^5}{240G}.\left(\frac{12}{bd^3}\right)^2$$

$$= \frac{3}{5}.\frac{P^2\ell}{AG}$$

Now External work done

$$W_{ext} = \frac{1}{2} \times P \times \Delta = \frac{1}{2}P\Delta$$

Equating external work done with strain energy due to bending and shear.

$$\frac{1}{2}P\Delta = \frac{P^2\ell^3}{6EI} + \frac{3}{5} \times \frac{P^2\ell}{AG}$$

$$\therefore \quad \Delta = \frac{P\ell^3}{3EI} + \frac{6}{5} \times \frac{P\ell}{AG}$$

It is to be noted that the first term at the right hand side is the deflection due to bending and the second term is due to shear.
Now re arranging the expression

$$\Delta = \frac{P\ell^3}{3EI}\left[1 + 3 \times \frac{6}{5} \times \frac{EI}{3AG\ell^2}\right]$$

$$= \frac{P\ell^3}{3EI}\left[1+3\times\frac{6}{5}\times\left(\frac{E}{G}\right)\frac{\frac{bd^3}{12}}{bd.\ell^2}\right]$$

$$= \frac{P\ell^3}{3EI}\left[1+3\times\frac{6}{5}\times\left(\frac{E}{G}\right)\frac{1}{12}.\frac{d^2}{\ell^2}\right]$$

$$= \frac{P\ell^3}{3EI}\left[1+\frac{3}{10}\times\left(\frac{E}{G}\right)\frac{d^2}{\ell^2}\right]$$

Significance of shear deformation

Let us now assume that $\frac{E}{G} = 2.5$ (a typical value for steel)

$$= \frac{P\ell^3}{3EI}\left[1+0.3\times 2.5.\frac{d^2}{\ell^2}\right]$$

$$= \frac{P\ell^3}{3EI}\left[1+0.75.\frac{d^2}{\ell^2}\right]$$

For $\ell = d$ (Short cantilever beam), the above expression gives

$$\Delta = \frac{P\ell^3}{3EI}\times 1.75$$

$$= 1.75\times\frac{P\ell^3}{3EI}$$

We know that the expression $P\ell^3/3EI$ is the end deflection of a cantilever under a point load P on it. As seen from the above expression, the total deflection due to shear and bending is 75% greater than that due to bending only.

Hence, shear deflection is very important in such short beams.
Again, for $\ell = 10d$ (For usual beams), the above expression gives.

$$\Delta = \frac{P\ell^3}{3EI}\left[1+0.75\times\frac{d^2}{10d^2}\right]$$

$$= \frac{P\ell^3}{3EI}[1+0.075]$$

$$= 1.075\times\frac{P\ell^3}{3EI}$$

Here the deflection due to shear is just 0.75% of the deflection due to bending. So, small deflection due to shear is significant for short beams and

in case of long beams, deflection due to shear can be neglected as it is too small.

Example # 3.13 A rectangular beam 20 *cm* × 40 *cm* ($b \times d$) is simply supported on a span of 6 *m* and carries a central load of 200 *kg*. Calculate the strain energy due to shear and bending. Neglect self-weight of the beam. Take $E = 2\times 10^6\ kg/cm^2$ and $G = 0.85\times 10^6\ kg/cm^2$.

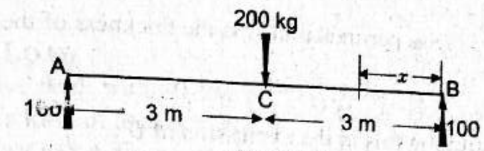


Fig. 3.19

Sol^n. As the beam is symmetrical, the reactions are equal and their magnitudes are 100 *kg*. each. Shear force F_x and bending moment M_x at x distance from B are:

$$F_x = 100\ kg$$

$$M_x = 100.x\ kg-m = 100\times 100.x\ kg-cm = 10000.x\ kg-cm$$

Strain energy due to shear force in given by

$$U_1 = K\times 2\int_0^3 \frac{V^2}{2AG}dx \qquad \text{where } K = 1.2$$

$$= \frac{1.2\times 2}{2AG}\int_0^{300}(100)^2 dx$$

$$= \frac{1.2\times 2\times 10000(300)}{2\times 20\times 40\times 0.85\times 10^6}$$

$$= 5.29\times 10^{-3}\ \text{kg-cm. } \underline{\text{Ans.}}$$

Strain energy due to bending is U_2

$$U_2 = 2\int_0^{\frac{\ell}{2}}\frac{M_x^2}{2EI}dx,\quad I = \frac{bd^3}{12} = \frac{20\times 40^3}{12} = 106666.67$$

$$= 2\int_0^{300}\frac{(10000x)^2 dx}{2\times 2\times 10^6\times 106666.67} = 2\int_0^{300}\frac{2\times(10000)^2}{2\times 2\times 10^6\times 106666.67}.\frac{x^3}{3}$$

$$= \frac{2(10000)^2\times 300^3}{2\times 2\times 10^6\times 106666.67\times 3}$$

$$= 4218.75\ kg\text{-}cm.\ \underline{\text{Ans.}}$$

Example # 3.14 **A solid circular shaft and thin walled circular tube made of the same material and having the same weight are stressed in torsion to the same maximum shear stress. What is the ratio of the amount of strain energies stored in the two shapes.**

Sol^n.

$$U_1 = \int_r \frac{T^2 dx}{2GJ} \quad \text{and} \quad U_2 = \int_v \frac{\tau^2}{2G} dv = \frac{\tau^2 v}{2G} = \frac{\tau^2}{2G} S\ell_2 t$$

$$= \frac{T^2 \ell_1}{2GJ} \qquad \text{(S is perimeter and t is the thickness of the tube)}$$

But, $T = \dfrac{\tau J}{r_1}$. Substituting this in the expression of U_1,

$$\therefore \quad U_1 = \frac{\tau^2 J^2}{r_1^2} \cdot \frac{\ell_1}{2GJ}$$

$$\text{or,} \quad \frac{U_1}{U_2} = \frac{\dfrac{\tau^2 J^2 \ell_1}{r_1^2 2GJ}}{\dfrac{\tau^2}{2G} S\ell_2 t} = \frac{J\ell_1}{r_1^2} \times \frac{1}{S\ell_2 t} \quad \text{..............................} (3.13)$$

But since weight are equal

$$\pi r_1^2 \ell_1 \rho = 2\pi r_2 t \ell_2 \rho \qquad \text{(Where ρ is density)}$$

$$\text{or,} \qquad t = \frac{r_1^2 \ell_1}{2 r_2 \ell_2}$$

Substituting the value of t in Eq. (3.13)

$$\frac{U_1}{U_2} = \frac{J\ell_1}{r_1^2} \frac{1}{S\ell_2} \times \frac{2r_2\ell_2}{r_1^2 \ell_1}$$

$$= \frac{\pi r_1^4 \ell_1}{r_1^2} \cdot \frac{1}{2\pi r_2 \ell_2} \times \frac{2 r_2 \ell_2}{r_1^2 \ell_1} = 1.2 \ \underline{\text{Ans.}}$$

Example # 3.15 Find the energy absorbed by an elastic circular rod subjected to a constant torque in terms of maximum shearing stress and the volume of the materials.

Sol^n. The shear stress is maximum at the periphery of rod as seen in the Fig (3.20) Let us consider a small annular section of thickness dx at a distance x from the centre of the cross section.

The shear stress acting on this is $\tau_{max} \cdot \dfrac{x}{r}$, Then,

$$U = \int \frac{\tau^2}{2G} . du = \int \frac{\tau_{max}}{2G} . \frac{x^2}{r^2} 2\pi x . dx \ell$$

$$= \frac{\tau_{max}^2}{2G} . \frac{2\pi\ell}{r^2} \int_0^r x^3 dx = \frac{\tau_{max}^2}{2G} . \frac{2\pi\ell}{r^2} . \frac{r^4}{4}$$

$$= \frac{\tau_{max}^2}{2G} \left(\frac{1}{2} Vol. \right) \underline{\text{Ans.}}$$

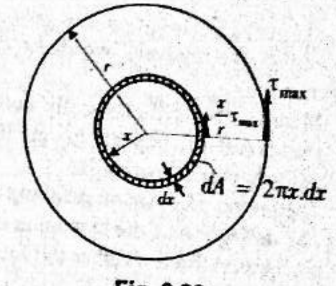


Fig. 3.20

3.9 IMPACT LOAD

So far, we have dealt with the load applied in the structures, which are static in nature. This kind of load is slowly applied, gradually increasing from zero to its maximum value, therefore, the load remains constant. There is another kind of load called dynamic or impact load which vary with time. Freely falling object or moving mass that strike the structure, load induced by traffic movement, wind gust, water waves are the examples of such load.

For the analysis of the impact load, let us consider a simple arrangement shown in Fig. (3.21). A collar of mass M, initially at rest falls from a height h on to a flange at the lower end of bar AB. When the collar falls down to the flange, its potential energy gets converted into kinetic energy. After the collar strikes the flange, its kinetic energy will be used (converted) as:

- Strain energy
- Heat energy
- Dissipates in causing localized plastic strain
- Some may still remain as kinetic energy for falling further down with flange as shown by dotted lines in Fig. (3.21–b)

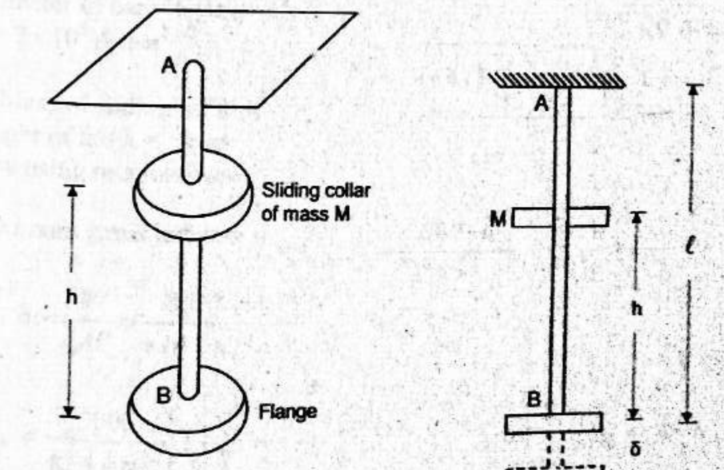


Fig. (a) Fig. (b)

Fig. 3.21

To simplify the analysis, we make the following assumptions for the above phenomenon.

(a) Materials behave elastically and no dissipation of energy takes place at the point of impact or at the supports owing to local inelastic deformation of materials.
(b) The inertia of a system resisting an impact is neglected.
(c) The deflection of the system is directly proportional to the magnitude of the applied force whether the force is applied statically or dynamically.

With these assumptions we can now apply the principle of conservation of energy to derive the expression for dynamic displacement, which is required to find stress in the flange. Referring Fig. (3.21), again let δ be the displacement of the flange after the collar strikes on it.

Potential energy of the collar $= W(h+\delta)$ .... (3.14)
where $(h+\delta)$ is the distance traveled by the collar and the flange.

Kinetic energy = Strain energy $= \dfrac{P^2\ell}{2AE} = \dfrac{AE\delta^2}{2\ell} \left(\because P = \dfrac{\delta AE}{\ell}\right)$ .... (3.15)

By principle of conservation of energy, (i.e. equating the above expressions)

or, $\delta^2\left(\dfrac{AE}{2\ell}\right) = W(h+\delta)$

or, $\delta^2 . \dfrac{AE}{2\ell} - W\delta - Wh = 0$

Solving for, δ

$$\delta = \frac{+W \pm \sqrt{W^2 - \dfrac{4AE}{2\ell}(-Wh)}}{\dfrac{2AE}{2\ell}}$$

$$= \frac{W\ell}{AE} + \left(\frac{W^2\ell^2}{A^2E^2} + \frac{4AE.Wh}{2\times\ell\times A^2E^2}\times\ell^2\right)^{\frac{1}{2}}$$

Put $\dfrac{W\ell}{AE} = \delta_{st}$

or, $\delta = \delta_{st} + \sqrt{\delta_{st}^2 + 2h.\delta_{st}}$

or, $\delta_d = \delta_{st}\left(1+\sqrt{1+\dfrac{2h}{\delta_{st}}}\right)$ .... (3.16)

Here δ is replaced by δ_d. It stands for dynamic deflection. For determination of stress due to impact, one has to find first the dynamic displacement (δ_d). The expression within the brackets of Eq. (3.16) is called impact factor (*I.F.*). Thus the equation can also be written as

$$\delta_d = \delta_{st} \times I.F.$$

(i) For suddenly applied load, $h = 0$

$$\therefore \delta_d = 2.\delta_{st}$$

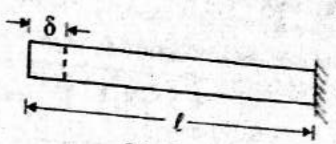


Fig. 3.22

(ii) For horizontal moving load, let us assume an arrangement as in Fig. (3.22)

Kinetic energy $= \dfrac{Mv^2}{2}$

Strain energy $= \dfrac{EA\delta^2}{2\ell}$ as before.

Equating these, we get. $\delta = \sqrt{\dfrac{Mv^2}{EA}}$

Example # 3.16 A bar 4 *m* long and 6 *cm* diameter hangs vertically and has a collar securely attached at the lower end. Find the maximum stress induced when (i) a weight 3000 *N* falls 10 *cm* on the collar (ii) a weight of 30000 *N* falls 1 *cm* on the collar. Take $E = 2\times10^5$ *N/mm*^{2}.

[T.U. 2056 Bhadra]

Given,
Length of bar $\ell = 4$ m
Diameter of bar = 6 cm
$E = 2\times10^5$ *N/mm*^{2}

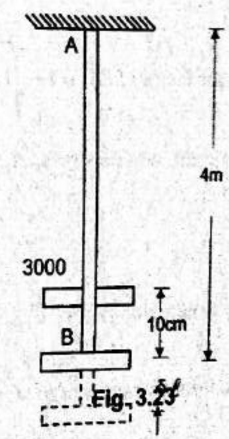


Fig. 3.23

(i) Mass of sliding collar $W = 3000$N
Height of fall $h = 10$ cm
Now using relation from above,

Maximum stress induced $\sigma_{max} = \dfrac{W}{A}\left(1+\sqrt{1+\dfrac{2h}{\delta\ell}}\right)$

But, $\delta\ell = \dfrac{W\ell}{AE} = \dfrac{3000\times4\times1000}{\pi/4\times60^2\times2\times10^5} = 0.0212$ *mm*

$$\sigma_{max} = \frac{3000}{\pi/4\times60^2}\left(1+\sqrt{1+\frac{2\times10\times10}{0.021}}\right)$$

$= 104.61$ *N/mm* ^{2} Ans.

(ii) Mass of sliding collar $W = 30\ 000N$
Height of fall $h = 1\ cm$
Now using relation from above,

Maximum stress induced $\sigma_{max} = \frac{W}{A}\left(1+\sqrt{1+\frac{2h}{\delta\ell}}\right)$

But, $\delta\ell = \frac{W\ell}{AE} = \frac{30000\times 4\times 1000}{\pi/4\times 60^2\times 2\times 10^5}$
$= 0.212\ mm$

$$\sigma_{max} = \frac{30000}{\pi/4\times 60^2}\left(1+\sqrt{1+\frac{2\times 10}{0.21}}\right) = 114.21\ N/mm^2$$

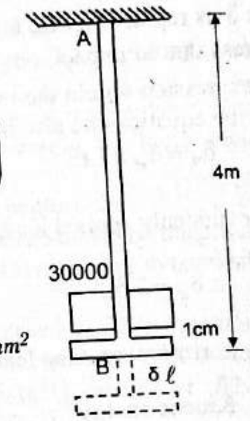


Fig. 3.24

Example # 3.17 A rounds prismatic steel bar of length 2 m and diameter 15 mm hangs vertically from a support at the upper end. A sliding collar of mass 20 kg drops from a height of 50 mm onto the flange at the lower end of the bar. Determine the impact factor and the maximum elongation, maximum tensile stress and maximum strain energy stored due to the impact. Take $E = 200\ GPa$

Sol^n.

Static deflection $\delta_{st} = \frac{W\ell}{AE} = \frac{20\times 9.81\times 2000}{200\times 10^3\times\pi\times\frac{15^2}{4}} = 0.011 N/mm^2$

Impact factor $(I.F.) = 1+\sqrt{1+\frac{2h}{\delta_{st}}} = 1+\sqrt{1+\frac{2\times 50}{0.011}} = 95.91$

Maximum tensile stress, $\sigma_t = \frac{W}{A}\times I.F. = \frac{20\times 9.81}{\pi\times\frac{15^2}{4}}\times 95.91$

$= 106.49\ N/mm^2$

Maximum elongation, $\delta_{max} = \frac{\sigma}{E}.\ell = \frac{106.49}{200\times 10^3}\times 2000 = 1.06\ mm$

Maximum strain energy stored, $U = \frac{\sigma^2_{max}}{2E}.V$

$$= \frac{(106.49)^2.\pi\times\frac{15^2}{4}\times 2000}{2\times 200\times 10^3}$$

$= 10019.82\ N-mm$
$= 1.0019\ kg-m$
$= 10.02\ Joules$ Ans.

Example # 3.18 A rectangular beam 200 mm × 100 mm is freely supported over a span of 2 m. A load of 10 kN is dropped on the middle of the beam from a height of 40 mm. Find maximum instantaneous defection and stress induced in the beam. Take $E = 1.1\times 10^4\ GPa$

[T.U. 2058 Shrawan]

Sol^n.
Given
Falling load, $W = 10kN$
Height of fall $h = 40\ mm = 0.4\ m$
Width of beam, $b = 200\ mm$
$= 0.2\ m$
Depth of beam $d = 100\ mm$
$= 0.1\ mm$

$\therefore I = \frac{bd^3}{12} = \frac{0.2\times 0.1^3}{12}$
$= 1.667\times 10^{-5}\ m^4$

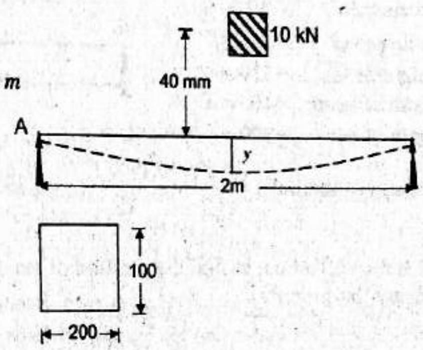


Fig. 3.25

Static deflection due to load W,

$\delta_{st} = \frac{W\ell^3}{48EI} = \frac{10\times 2^3}{48\times 1.1\times 10^4\times 10^3\times 1.667\times 10^{-5}}$
$= 0.009\ m = 9\ mm$

Impact $I.F. = 1+\sqrt{1+\frac{2h}{\delta\ell}}$

$= 1+\sqrt{1+2\times\frac{40}{9}}$

$= 4.134$

(i) $\delta_d \doteq \delta_{st}\times I.F.$
$= 9\times 4.13 = 37.17\ mm$ Ans.

(i) Instantaneous stress developed σ_1

$$\sigma_1 = \frac{M}{Z} = \frac{WL}{4\times Z} = \frac{10\times 2}{4\times\frac{1}{6}\times 0.2\times 0.1^2} = 15000\ kN/m^2$$

$\therefore\quad \sigma_1 = I.F.\times\sigma = 4.134\times 15000 = 62010\ kN/m^2$ Ans.

Example # 3.19 **Find instantaneous maximum deflection and bending stress for a 100 mm × 100 mm simply supported steel beam of 3 m span when it is struck at a distance of 2 m from the left support by a 200 N weight falling from a height of 150 mm above the top of the beam. Take $E = 200\times10^5\ N/mm^2$. Also find the maximum deflection and the bending stress when the same load is suddenly applied.** [T.U. 2057 Bhadra]

Sol^n.

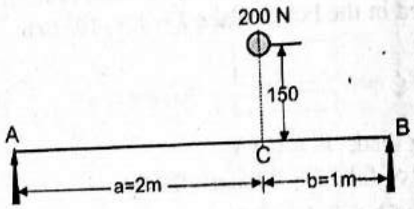


Fig. 3.26

Here,

Given span $= 3\ m$

Falling load $= 200\ N$

Height of fall $= 150\ mm$

Width of beam $= 100\ mm\ (b)$

Depth of beam $= 100\ m\ (d)$

Moment of inertial $I = \dfrac{bd^3}{12} = \dfrac{0.1\times0.1^3}{12} = 8.333\times10^{-6} m^4$

For static deflection, to find the position of maximum deflection from left end, use the formula

$$x = \sqrt{\frac{\ell^2 - b^2}{3}} \quad or \quad \sqrt{\frac{a^2+2ab}{3}}$$

$$= \sqrt{\frac{3^2-1^2}{3}}$$

$= 1.63\ m$ from left end.

Maximum deflection $\delta_{max} = \dfrac{Wb\left(\ell^2-b^2\right)^{3/2}}{9\sqrt{3}EI.\ell}$

$$= \frac{200\times1(3^2-1^2)^{\frac{3}{2}}}{9\sqrt{3}\times200\times10^5\times10^6\times8.333\times10^{-6}\times3}$$

$= 5.807\times10^{-7}\ m$

$= 5.807\times10^{-4}\ mm$

$= 0.0005\ mm$ Ans.

Now, Impact factor, $I.F. = 1+\sqrt{1+\dfrac{2h}{\delta\ell}}$

$$= 1+\sqrt{1+2\times\frac{150}{0.0005}}$$

$= 776$

Bending stress developed at C,

$$\sigma = \frac{M}{Z} = \frac{R_B\times1}{Z} = \frac{133.33\times1}{\frac{1}{6}\times0.1\times0.1^2} = 800\ kN/m^2$$

$= 800\ kN/m^2$ Ans.

Now, instantaneous stress developed at C.

$\sigma_{i1} = I.F.\times\sigma$

$= 776\times800\ kN/m^2$

$= 620800\ kN/m^2$ Ans.

Instantaneous deflection (Maximum)

$\delta_{i1} = I.F.\times\delta_1$

$= 776\times0.0005\ mm$

$= 0.39\ mm$ Ans.

When the same load is suddenly applied, $h=0$, $I.F.=2$

Instantaneous stress developed a C.

$\delta_{i2} = 2\times.0005$

$= .001\ mm$

Example # 3.20 **Find the instantaneous maximum deflection and bending stress for the 50 mm × 50 mm steel beam of 1 m span and simply supported when struck at the mid span by a 150 N weight falling from a height of 75 mm above the top of the beam. Take $E = 200GPa$. Also, find the maximum deflection and bending stress when the same load is suddenly applied.**

Sol^n. Strain energy stored = Internal work done = W_{int}

$$=2\int_0^{\ell/2}\frac{M_x^2dx}{2EI} = 2\int_0^{\ell/2}\frac{(75x)^2}{2EI}dx$$

$$= \frac{2\times75^2}{2\times200\times10^3\times50\times\frac{50^3}{12}}\cdot\frac{(500)^3}{3}$$

$= 2.25$

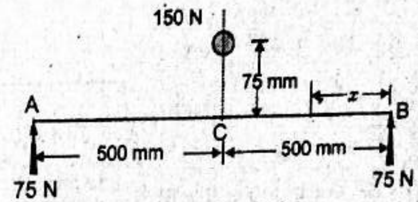


Fig. 3.27

$W_{ext} = \dfrac{1}{2}\times150\times\delta_{st} = 75\delta_{st}$

$W_{ext} = W_{int}$

or, $\delta_{st} = \dfrac{2.25}{75} = 0.03\ mm$ Ans.

Impact factor $I.F. = 1+\sqrt{1+\dfrac{2h}{\delta st}} = 1+\sqrt{1+\dfrac{2\times 75}{.03}} = 71.72$

Instantaneous deflection δ_{ist} or $\delta_d = \delta_{st} \times I.F. = 0.03\times 71.72 = 2.15\ mm$

Bending stress $\sigma_{st} = \dfrac{M}{I}.y = \dfrac{75\times 500\times 25}{50\times\dfrac{50^3}{12}} = 1.8\ MPa$ Ans.

For suddenly applied load,

Impact factor $= 1+\sqrt{1+\dfrac{2h}{\delta st}}$ where, $h=0$

$= 2$

$\delta_{max} = 0.03\times 2 = 0.06\ mm$ Ans.

$\sigma_{max} = 1.8\times 2 = 3.6\ MPa$ Ans.

Example # 3.21 An unknown weight falls through 2 *cm* on a collar rigidly attached to the lower end of a vertical bar 3 *m* long and 7 cm^2 in cross-section. If the maximum instantaneous extension is 3 *mm*, what is the corresponding stress and value to the unknown weight? Take $E = 2\times 10^6\ kg\ cm^2$.

Sol^n. We have,

Instantaneous displacement

$$\delta d = \delta st\left(1+\sqrt{1+\frac{2h}{\delta st}}\right) \quad \text{where } \delta st = \frac{W\ell}{AE} = \frac{W\times 3\times 1000}{7\times 100\times 2\times 10^6\times\dfrac{10}{100}}$$

$$= 2.14\times 10^{-5} W\ mm.$$

But $\delta d = 3\ mm$ given,

$$\therefore\quad 3 = 2.14\times 10^{-5} W\left(1+\sqrt{1+\frac{2\times 2\times 10}{2.14\times 10^{-5} W}}\right)$$

or, $140000 = W + W\sqrt{1+\dfrac{1869159}{W}}$

or, $140000 = W + \dfrac{W}{\sqrt{W}}\sqrt{W+1869159}$

Solving by trial and error method, $W = 9119.5\ kN$ Ans.

Example # 3.22 An unknown weight falls through a height of 10 *mm* on a collar rigidly attached to the lower end of a vertical bar 5 *m* long and 600 mm^2 in section. If the maximum extension of the rod is to be 2 *mm*, what is the corresponding stress and magnitude of the unknown weight? Take $E = 200\ GN/m^2$.

$h = 10\ mm = 10\times 10^{-3}\ m$

$\ell = 5\ m$

$A = 600\ mm^2$

$\delta d = 2\times 10^{-3}\ m$

For instantaneous stress σ,

$$E = \frac{\sigma}{\varepsilon} = \frac{\sigma}{\delta\ell/\ell}$$

or, $\sigma = E.\dfrac{\delta\ell}{\ell} = \dfrac{200\times 10^9\times 2\times 10^{-3}}{5}$

$= 80\times 10^6\ N/m^2$ Ans.

For unknown weight, we have,

$$\delta d = \delta st\left(1+\sqrt{1+\frac{2h}{\delta st}}\right), \quad \text{where}$$

$$\delta st = \frac{W\ell}{AE} = \frac{W\times 5}{600\times 10^{-6}\times 200\times 10^9} = 4.17\times 10^{-8} W$$

Now,

$$2\times 10^{-3} = 4.17\times 10^{-8} W\left(1+\sqrt{1+\frac{2\times 10\times 10^{-3}}{4.17\times 10^{-8} W}}\right)$$

or, $\dfrac{2\times 10^{-3}}{4.17\times 10^{-8}} = W+\sqrt{W^2+\dfrac{2\times 10\times 10^{-3} W}{4.17\times 10^{-8}}}$

or, $(47961.63 - W) = \sqrt{W^2 + 479616.31W}$

or, $(47961.63 - W)^2 = W^2 + 479616.31W$

or, $(47961.63)^2 - 2\times 47961.63W + W^2 = W^2 + 479616.31W$

or, $W = \dfrac{(47961.63)^2}{(2\times 47961.63 + 479616.31)} = 3997\ N$ Ans.

Example # 3.23 A lift carrying a total weight of 4 *tonnes* moves downward with a constant velocity of 1 *m/sec*. What is the maximum stress produced on the steel rope that supports the lift when its upper end is suddenly stopped. The free end of the rope at the moment of stoppage is 30 *m* and the cross

sectional area of the rope is 12 *sq. cm*. Take Modulus of Elasticity of the steel rope as 2×10^6 *kg/cm*^{2}.

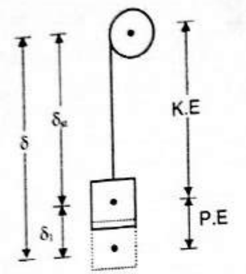


Fig. 3.28

Sol^n.

Let W be the weight of the lift that is moving with a downward a velocity v. After the upper end is stopped the lift travel a distance δ. But actually the lift travel the distance lesser than δ. As the cable was already elongated by amount δst due to the load W. The actual displacement would thus be $\delta_1 = \delta - \delta st$.

Now equating the energies before and after stoppage, we get

$$\frac{Wv^2}{2g} + W(\delta - \delta st) + \frac{EA\delta st^2}{2\ell} = \frac{EA\delta^2}{2\ell}$$

(Here *K.E.* of cable is neglected as it is small)

since, $\delta st = \frac{W\ell}{EA}$ and $W = \frac{EA\delta st}{\ell}$

$$\frac{Wv^2}{2g} + \frac{EA\delta st}{\ell}(\delta - \delta st) = \frac{EA}{2\ell}(\delta^2 - \delta st^2)$$

or, $$\frac{Wv^2}{2g} = -\frac{EA}{2\ell}(2\delta st.\delta - 2\delta st^2 - \delta^2 + \delta st^2)$$

$$= -\frac{EA}{2\ell}(2\delta st.\delta - \delta st^2 - \delta^2) = \frac{EA}{2\ell}(\delta - \delta st)^2$$

or, $$(\delta - \delta st)^2 = \frac{Wv^2}{2g} \times \frac{2\ell}{EA} = \frac{Wv^2\ell}{gEA}$$

$$\delta = \sqrt{\frac{Wv^2\ell}{gEA}} + \delta st$$

The stress produced is thus $\sigma = \frac{E\delta}{\ell} = \frac{W}{A}\left(1 + \sqrt{\frac{v^2EA}{gW\ell}}\right)$

$$\therefore \quad \frac{W}{A}\left(1 + \sqrt{\frac{v^2EA}{gW\ell}}\right) = \frac{4 \times 1000}{12 \times 10^{-4}}\left(1 + \sqrt{\frac{1^2 \times 2 \times 10^6 \times 10^4 \times 12 \times 10^{-4}}{9.81 \times 4 \times 1000 \times 30}}\right)$$

$$= 1.83 \times 10^7 \; kg/m^2 \; \underline{\text{Ans}}$$

Example # 3.24 A block of mass is moving with a velocity v_o hits squarely the prismatic member AB at its midpoint C. Determine (a) equivalent static load (b) maximum stress σ_m in the member and (c) maximum deflection at point.

(a) <u>Equivalent static load:</u> Maximum strain energy of the member is equal to the kinetic energy of the block before impact.

We have

$$U = \frac{1}{2} m v_0^2 \;\ldots\ldots\ldots\; (3.17)$$

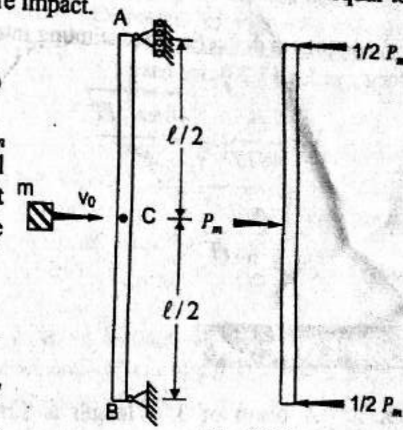


Fig. 3.29

On the other hands, expressing U_m as the work, equivalent horizontal static load as it is slowly applied at the midpoint C of the member, we can write

$$U_m = \frac{1}{2} P_m \, y \;\ldots\ldots\; (3.18)$$

Where y is the deflection of C corresponding to the static load P_m from deflection of beam we can write

$$y = \frac{P_m \ell^3}{48 EI} \;\ldots\ldots\ldots\; (3.19)$$

Substituting y from Eq. (3.19) to (3.18), we get

$$U_m = \frac{1}{2} \times P_m \times \frac{P_m \ell^3}{48 EI}$$

$$= \frac{1}{2} \times \frac{P_m^2 \ell^3}{48 EI}$$

Solving for P_m and recalling Eq. (3.17), we find that the static load equivalent to the given impact loading is

$$P_m = \sqrt{\frac{96 U_m EI}{\ell^3}} = \sqrt{\frac{48 m v_0^2 EI}{\ell^3}} \;\ldots\ldots\ldots\ldots\; (3.20)$$

(b) Maximum stress: The maximum stress, therefore, occurs in a transverse section through C and is equal to

$$\sigma_m = \frac{M_{max} \cdot \bar{y}}{I} = \frac{P_m \ell}{4I} \cdot \bar{y}$$

$$= \sqrt{\frac{48 m v_0^2 EI}{\ell^3}} \times \frac{\ell}{4I} \cdot \bar{y}$$

$$= \sqrt{\frac{3 m v_0^2 E\ell}{I(\ell/\bar{y})^2}}$$

(c) Maximum deflection: Substituting into Eq. (3.19) the expression obtained for P_m in Eq. (3.20), we have

$$y = \frac{\ell^3}{48\,EI} \times \sqrt{\frac{48 m v_0^2 EI}{\ell^3}}$$

$$= \sqrt{\frac{m v_0^2 \ell^3}{48\,EI}}$$

3.10 Exercises

Ex. 1 A beam of 3 m length is simply supported at each end and is subjected to a couple of 9 kN-m at a point B, 2 m from the left end shown in Fig. (3.30). Determine the slope at B. Take $EI = 30\ kN\text{-}m^2$.

Ans. $\theta = 0.1$ rad.

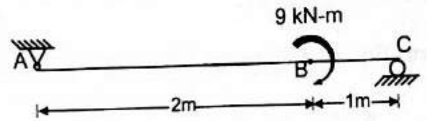


Fig. 3.30

Ex. 2 A metal specimen 1.5 m^2 in cross section stretches 0.05 mm over 10 cm gauge length under an axial load of 40 kN. Calculate the strain energy stored in the specimen at this point. If the load at the elastic limit for specimen is 60 kN. Calculate the elongation at the elastic limit and resilience.

Ans. 533.33 GN/m^2, 0.0075 mm

Ex. 3 A 120 cm in length is subjected to an axial pull such that the maximum stress is equal to 150 MN/m^2. Its area of cross section is 2 cm^2 over a length of 100 cm and for middle 20 cm length it is only 1 cm^2. If $E = 200\ GN/m^2$. Calculate the strain energy stored in bar.

Ans.. 3.9375 Joule

Ex. 4 A wagon weighing 80 kN is attached to wire-rope and moving down an incline at speed of 0.5 m/s when rope jams and the wagon is suddenly brought to rest. If the length of the rope is 60 m at the time of stoppage. Calculate maximum instantaneous stress and maximum instantaneous elongation produced. Diameter of rope = 50 mm and $E = 200\ GN/m^2$.

Ans.. 58.833 MN/m^2, 1.764 mm

Ex. 5 An unknown weight falls through 15 mm an a collar rigidly attached to the lower end of a vertical bar 4 m long and 800 mm^2 in section. If the maximum extension of the rod is to be 2 mm, what is the corresponding stress and magnitude of unknown weight ?

Ans. $100 \times 10^6\ MN/m^2$, 80 N

Ex. 6 A 2 mm long beam rectangular in section 40 mm × 60 mm is supported an rigid supports at its ends. If it is struck at the center by a 500 kg mass falling through a height of 80 mm.
Find
(i) Instantaneous stress developed.
(ii) The instantaneous strain energy stored in beam.

Ans. 654.507 MN/m^2, 588.23 J

4

VIRTUAL WORK METHOD

4.1 WORK AND COMPLEMENTARY WORK

We know that a structure deforms when a force is applied on it. The product of the force and the deformation gives the work done by the force. The work done is represented graphically by the area under the force deformation curve line shown in Fig.(4.1).

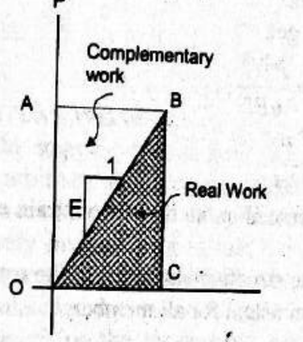


Fig. 4.1

The area above the line OB is complementary work such that for a linear system, the sum of work and the complementary work is equal to zero.

4.2 DEFLECTION BY METHOD OF REAL WORK

As described earlier, the external work done (W) which is the input energy in the structure is stored in it as the internal strain energy (U). By the principle of conservations of energy, these two quantities (W and U) can be equated to find the deflection of a structure. The procedure is illustrated in the following examples. This method of finding the deflection is also called as "Real work method" since work done by the actual loads are considered. This method however faces problem when there are several loads applied in the structure and deflection is desired not at the point of application of the loads.

Example # 4.1 Find the deflection at B in the cantilever by real work method.

Sol^n.

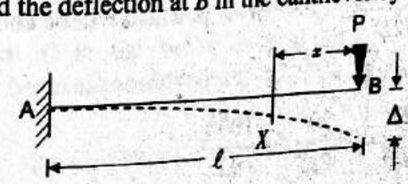


Fig. 4.2

Real work done by the load P

$$W=\frac{P}{2}.\Delta \qquad (\frac{P}{2} \text{ is the average load})$$

Strain energy stored

$$U=\int_0^\ell \frac{M^2 dx}{2EI}$$

$$=\int_0^\ell \frac{(Px)^2 .dx}{2EI}$$

$$=\frac{P^2}{2EI}.\int_0^\ell x^2 dx$$

$$=\frac{P^2\ell^3}{6EI}$$

Equating W and U, we get.

$$\frac{P}{2}.\Delta=\frac{P^2\ell^3}{6EI}$$

$$\therefore \Delta=\frac{P\ell^3}{3EI}$$

Note: This is also illustrated in the chapter of strain energy method.

Example # 4.2 For the structure shown, find the rotation of C by the method of real work. Take EI constant for all members.

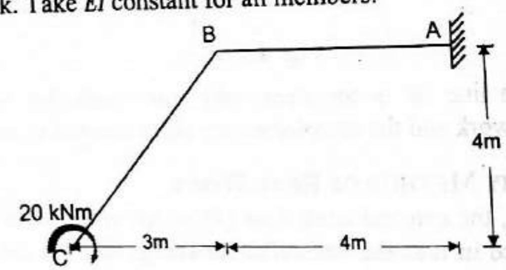


Fig. 4.3

Sol^n. The total strain energy stored in the structure is the sum of the energies stored in the two members AB and BC as.

$$U_{AC}=U_{AB}+U_{BC}$$

$$U_{BC}=\int_0^\ell \frac{M^2 dx}{2EI}, \qquad M=20 \text{ which will be constant along } CB.$$

$$\ell=CB=\sqrt{3^2+4^2}=5$$

$$=\int_0^5 \frac{20^2}{2EI}dx=\frac{20^2}{2EI}.5=\frac{1000}{EI}$$

$$U_{AB}=\int \frac{M^2 dx}{2EI} \qquad \text{(Here also } M=20 \text{ from } B \text{ to } A)$$

$$=\int_0^4 \frac{(20)^2 dx}{2EI}=\frac{20^2}{2EI}.4=\frac{800}{EI}$$

$$U_{AC}=\frac{1000}{EI}+\frac{800}{EI}=\frac{1800}{EI}$$

External work done $=\frac{1}{2}M.\theta=\frac{1}{2}\times 20\times\theta$

From the conservation of energy, $\frac{1}{2}M.\theta=U_{AB}$

or, $\quad \frac{1}{2}\times 20\times\theta=\frac{1800}{EI}$

or, $\quad \theta=\frac{180}{EI}$ rad. Ans.

4.3 Principle of Virtual Work

It is always possible to imagine that a real structure system in static equilibrium is given an arbitrary displacement consistent with the boundary condition. During the process, the real system of forces undergoes imaginary displacement. Alternatively imaginary or virtual forces in equilibrium with the given system can be given real, kinematically admissible displacement. In either case one can formulated virtual work, which is the product of real force and imaginary displacement or the product of imaginary force and real displacement.

If a rigid body in equilibrium is given an arbitrarily virtual displacement, the total virtual work done by the external forces must be zero. The internal forces need not be considered because; the distances between any two points in a rigid system must remain constant. But in the case of deformable body, the virtual work done may be caused by real forces acting through virtual displacement. According to the principle of conservation of energy, the total virtual work done by the internal and external forces on a deformable system must be zero. By applying the conservation of energy one can find the deformation of structures, which will be illustrated in the following section.

4.4 Application of Virtual Work

The principle of virtual work is used to derive an important relation used for finding deflection of structures. Let us assume that a virtual force V^* is initially applied at C in the beam in Fig. (4.4-a). This force causes deformation in the beam as indicated by the dotted line.

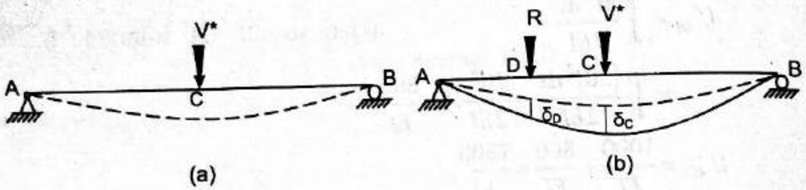


Fig. 4.4

Now, let a real force (R) is applied at D on the same beam. The force (R) brings additional displacement in the beam and the deflected curve of the beam is indicated by the continuous line in Fig. (4.4-b). The displacement, δ_c and δ_d are brought by force R. Our interest now is to find δ_c. The force V^* rides off and the external virtual work done is given by $W_{ext} = V^* . \delta_c$

The internal work done by deformation is the work done by stress resultants (Moment in this case as deflection due to axial force and $S.F.$ are neglected). Thus W_{int} is the work done by moment M^* caused by the unspecified virtual force V^* acting through the bending deformation (angle change $d\theta$) produced by real force R.

$$W_{\text{int}} = \int_0^{\ell} M^* d\theta \qquad \text{But } d\theta = \frac{M}{EI} dx$$

or, $$W_{\text{int}} = \int_0^{\ell} \frac{M^* M dx}{EI}$$

Now by principle of virtual work,

$$W_{ext} = W_{\text{int}}$$

or, $$V^* \times \delta_c = \int_0^{\ell} \frac{MM^* dx}{EI}$$

If $V^* = 1$, then $$\delta_c = \int_0^{\ell} \frac{MM^* dx}{EI} \quad \text{.......................... (4.1)}$$

This indicates that under the application of unit virtual force, we get an expression for deflection due to the real load system. The unit external force can be in the form of either a force or a moment depending upon the form of external displacements that are to be determined. In the above expression, it is to be noted that M is the internal force (moment) brought by real force R and M^* is the internal force brought by the virtual force V^*. The expression is useful to find deflection by unit load method. The process to find the deflection by this method is thus described in the following steps.

(i) Determine the internal force (M in the above expression) due to the application of external forces.

(ii) Remove all the external forces and apply unit load in the direction of displacement to be computed and find the internal force (M^* in the above expression)

(iii) Compute $\int \frac{MM^*}{EI} .dx$ to which is equal to the deflection.

The following problems illustrate the detail procedure.

4.5 DEFLECTION OF TRUSSES

The internal forces in the trusses are the member forces F. So the expression for deflection is obtained by replacing M and M^* by F and F^*. The expression thus becomes

$$\delta = \sum \frac{FF^* \ell}{AE} \quad \text{.....................(4.2)}$$

Example # 4.3 Find the defection and slope at point B of the cantilever beam using unit load method.

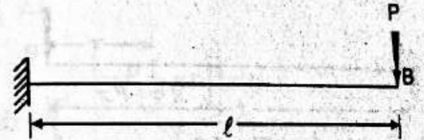
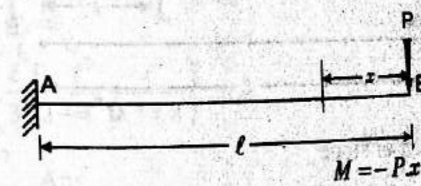


Fig. 4.5

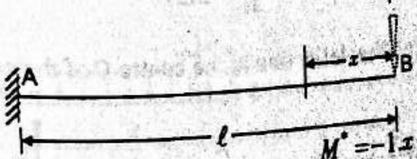
Sol^n.

(a) <u>For deflection</u>

<u>Step 1</u>. Find the expression for moment due to the external load (M) for the beam.



<u>Step 2</u>. Remove the external load P and apply a unit load (1^*) at B (the point where deflection is being computed) and determine expression for moment (M^*)



Step 3. Use expression for deflection, i.e.

$$\delta = \int_0^{\ell} \frac{MM^* dx}{EI}$$

$$= \int_0^{\ell} \frac{(-Px)(-x)dx}{EI} = \frac{P}{EI}\int_0^{\ell} x^2 dx = \frac{P}{EI}\left[\frac{x^3}{3}\right]_0^{\ell}$$

$$\delta = \frac{P\ell^3}{3EI} \quad \underline{\text{Ans.}}$$

(The direction of the deflection is downward as of the unit load.)

b) For Slope

To find slope, similar procedure as above is followed but the unit load to be applied here will be moment. The applied moment produces slope (to be determined) whereas applied force produces deflection.

Step 1. Find the expression for the moment due to the external load (as above)

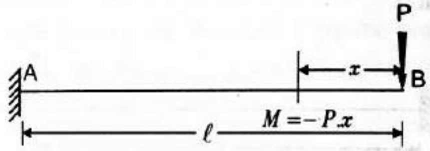
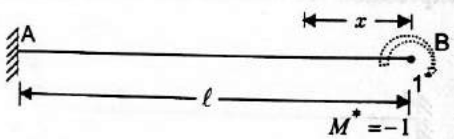


Fig. 4.6

Step 2. Remove the external load P and apply a unit moment (not force) at point B and find the expression for moment in the beam.



Step 3. Use expression of slope

$$\theta_B = \int_0^{\ell} \frac{MM^* dx}{EI}$$

$$= \int_0^{\ell} \frac{(-Px)(-1)dx}{EI} = \left[\frac{-Px^2}{2EI}\right]_0^{\ell} = \frac{P\ell^2}{2EI} \quad \underline{\text{Ans.}}$$

Example # 4.4 Find the deflection at the centre C of the cantilever beam.

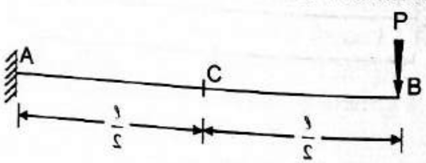
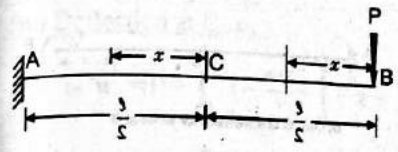
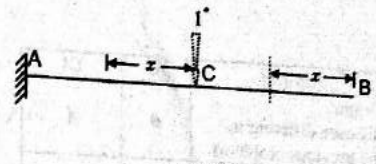


Fig. 4.7

Sol^n. This is slightly different to the previous problem. The deflection required is at the centre of the beam. So we need to write the expressions for M and M^* for the sections AC and CB instead of writing for the whole span of the beam. Consequently, the limit of integration also changes.



For M



For M*

The expressions for M and M^* are shown in the following tabular form for convenience. Now,

Segment	BC	CA
Origin	B	C
Limits	$0 \to \frac{\ell}{2}$	$\frac{\ell}{2} \to \ell$
M	$-Px$	$-P\left(\frac{\ell}{2}+x\right)$
M^*	$-0x$	$-1x$

$$\delta_c = \int_{BC} \frac{MM^*}{EI}.dx + \int_{CA} \frac{MM^*}{EI}.dx$$

$$= \int_0^{\ell/2} \frac{(-Px).0}{EI}.dx + \int_{\ell/2}^{\ell} \frac{\left[-P\left(\frac{\ell}{2}+x\right)\right](-x)}{EI}.dx$$

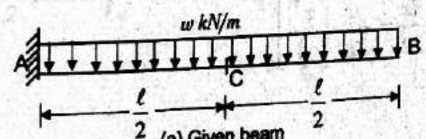
$$= 0 + \frac{P}{EI}\int_{\ell/2}^{\ell}\left(\frac{\ell x}{2}+x^2\right)dx$$

$$= \frac{P}{EI}\left[\int_0^{\ell/2}\frac{\ell}{2}x.dx + \int_0^{\ell/2} x^2 dx\right]$$

$$= \frac{P}{EI}\left(\frac{\ell}{3}.\frac{1}{2}.\frac{\ell^2}{4} + \frac{\ell^3}{3\times 8}\right)$$

$$= \frac{5P\ell^3}{48EI} \quad \underline{\text{Ans.}}$$

Example # 4.5 Find the slope and deflections at points B and C in the cantilever beam loaded with uniformly distributed load of w as shown.

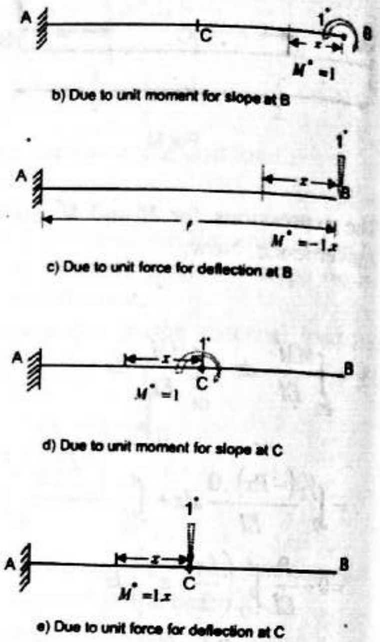


(a) Given beam

Fig. 4.8

Solⁿ. Let us draw the diagrams to find the expression for M and M^* as in the previous examples.

Segment	BC	CA
Origin (Chosen different to the previous problem)	B	A
Limits	$0 \rightarrow \frac{\ell}{2}$	$\frac{\ell}{2} \rightarrow 0$
M	$-\frac{wx^2}{2}$	$-\frac{wx^2}{2}$
M^* due to units moment at B Fig. (b)	-1	-1
M^* due to unit load at B Fig. (c)	$-x$	$-x$
M^* due to unit moment at C Fig. d)	0	-1
M^* due to unit load at C Fig. (e)	0	$-\left(x-\frac{\ell}{2}\right)$



(i) <u>Slope a B θ_B</u>

$$\theta_B = \int_0^\ell \frac{MM^* \, dx}{EI} = \int_0^\ell \left(\frac{-wx^2}{2}\right)(-1).\frac{dx}{EI}$$

$$= \frac{w}{2EI}\left[\frac{x^3}{3}\right]_0^\ell = \frac{w\ell^3}{6EI} \quad \underline{\text{Ans.}}$$

(ii) <u>Deflection at B, y_B</u>

$$y_B = \int_0^\ell \left(\frac{wx^2}{2}\right)(-x)\frac{dx}{EI} = \frac{w}{2EI}\left[\frac{x^4}{4}\right]_0^{\ell/2}$$

$$= \frac{w\ell^4}{8EI} \quad \underline{\text{Ans.}}$$

(iii) <u>Slope at C, θ_c</u>

$$\theta_c = 0 + \int_0^{\ell/2}\left(\frac{wx^2}{2}\right)(-1)\frac{dx}{EI} = \frac{w}{2EI}\left[\frac{x^3}{3}\right]_0^{\ell/2} = \frac{7}{48}\cdot\frac{w\ell^3}{8EI}$$

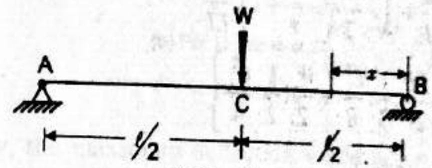
(iv) <u>Deflection at C, y_c</u>

$$y_c = 0 + \int_0^{\ell/2}\left(-\frac{wx^2}{2}\right)\left(-x+\frac{\ell}{2}\right)\frac{dx}{EI}$$

$$= \frac{w}{2EI}\left[\frac{x^4}{4} - \frac{\ell x^3}{6}\right]^{\ell/2}$$

$$= \frac{7}{384}\cdot\frac{w\ell^4}{EI} \quad \underline{\text{Ans.}}$$

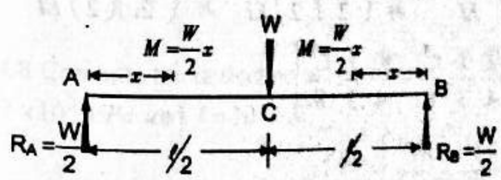
Example # 4.6 Calculate the slope at B and deflection at C in the simply supported beam loaded with central point load shown below.



a) Given beam

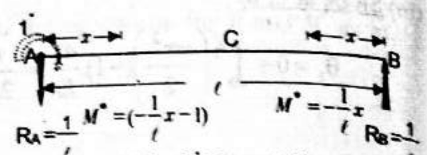
Fig. 4.9

Solⁿ. Let us draw diagram to find the expressions for M and M^*. Let us also write the expressions for M and M^* systematically as in the table given below and compute the required expressions.

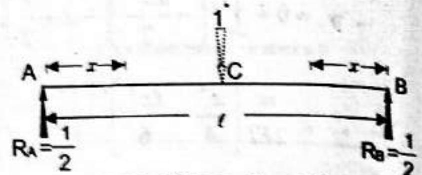


(b) For M

Segment	BC	CA
Origin	B	A
Limits	$0 \to \frac{\ell}{2}$	$0 \to \frac{\ell}{2}$
M Fig (b)	$\frac{Wx}{2}$	$\frac{Wx}{2}$
M' Fig. (c)	$-\frac{x}{\ell}$	$-\left(\frac{x}{\ell}+1\right)$
M' Fig. (d)	$\frac{x}{2}$	$\frac{x}{2}$



(c) For M' (slope at C)



(d) For M' (deflection at C)

(i) Slope at $B = \theta_B$

$$\theta_B = \int_0^{\ell} \frac{MM^* dx}{EI}$$

$$= \int_0^{\ell/2} \left(\frac{Wx}{2}\right)\left(\frac{x}{\ell}\right)\frac{dx}{EI} + \int_0^{\ell/2} \frac{Wx}{2}\left(-\frac{x}{\ell}+1\right)\frac{dx}{EI}$$

$$= \frac{1}{EI}\left[\frac{W}{2\ell}\cdot\frac{1}{3}\cdot\frac{\ell^3}{8} - \frac{W}{2\ell}\cdot\frac{1}{3}\cdot\frac{\ell^3}{8} + \frac{W}{2}\cdot\frac{1}{2}\cdot\frac{\ell^2}{4}\right]$$

$$= \frac{1}{EI}\left[\frac{W\ell^2}{48} - \frac{W\ell^2}{48} + \frac{W\ell^2}{16}\right]$$

$$= \frac{W\ell^2}{16EI} \quad \underline{\text{Ans.}}$$

(ii) Deflection at $C = \delta_c$

$$\delta_c = \int_0^{\ell/2} \frac{MM^* dx}{EI} = \int_0^{\ell/2}\left(\frac{Wx}{2}\right)\left(\frac{x}{2}\right)\frac{dx}{EI} + \int_0^{\ell/2}\left(\frac{Wx}{2}\right)\left(\frac{x}{2}\right)\frac{dx}{EI}$$

$$= \frac{1}{EI}\left[\frac{W}{4}\cdot\frac{1}{3}\cdot\frac{\ell^3}{8} + \frac{W}{4}\cdot\frac{1}{3}\cdot\frac{\ell^3}{8}\right]$$

$$= \frac{1}{EI}\left[\frac{W\ell^3}{96} + \frac{W\ell^3}{96}\right]$$

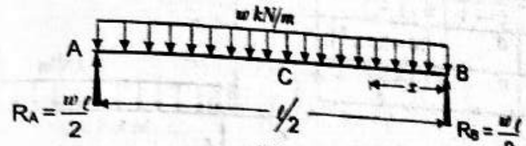
$$= \frac{W\ell^3}{48EI} \quad \underline{\text{Ans.}}$$

The procedure adopted in this problem indicates that the origins for the segment should not necessarily be from one point only. It is convenient to

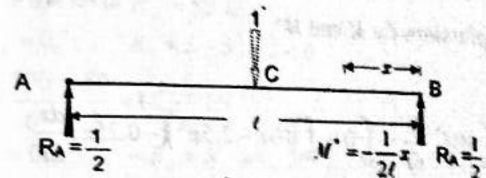
choose the origin at B for segment CB and A for segment AC. Alternatively; one can also choose B as a origin for both the segments, (BC and CA). In that case, the expressions for M will be different.

Example # 4.7 Find the maximum deflection of the simply supported beam of span ℓ loaded with uniformly distributed load of w on its entire span.

Sol^n.



a) For M



(b) For M' (slope at C)

Fig. 4.10

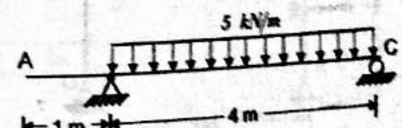
Now obviously, the maximum deflection would occur at C and its value is given by

$$\delta_c = \int_0^{\ell} MM^* \frac{dx}{EI} = \frac{1}{EI}\int_0^{\ell}\left(\frac{w\ell x}{2} - \frac{wx^2}{2}\right)\left(\frac{x}{2}\right)dx$$

$$= \frac{1}{EI}\left[\frac{w\ell}{2\times 2}\cdot\frac{\ell^3}{3} - \frac{w}{4}\cdot\frac{\ell^4}{4}\right]$$

$$= \frac{5w\ell^4}{48EI} \quad \underline{\text{Ans.}} \quad \frac{5wl^4}{384EI}$$

Example # 4.8 Compute the deflection at A due to the loading shown below. Take $E = 207\times10^3\ MPa$ and $I = 10^{-4}\ m^4$

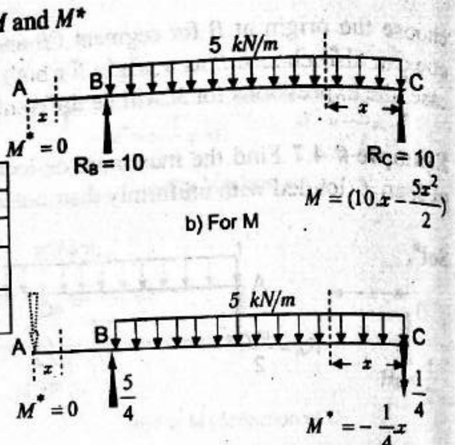


a) Given beam

Fig. 4.11

Sol^n. Let us draw the diagram for M and M^*

Segment	AB	CB
Origin	A	C
Limits	$0 \to 1$	$0 \to 4$
M Fig. (b)	0	$(10 - 2.5x^2)$
M^* Fig. (c)	0	$(-1/4)\times x$



b) For M

c) For M^*

Writing the expressions for M and M^*,
we have,

$$\delta_a = \int_0^{\ell} MM^* \frac{dx}{EI} = \int_0^1 0 + \int_0^4 (10x - 2.5x^2)(-0.25x)\frac{dx}{EI}$$

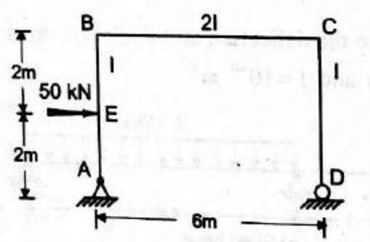
$$= \int_0^4 (-2.5x^2 + 0.625x^3)\frac{dx}{EI}$$

$$= \frac{1}{EI}\left(-2.5 \times \frac{4^3}{3} + 0.625 \times \frac{4^4}{4}\right)$$

$$= -\frac{13.33}{EI} = \frac{-13.33 \times 1000}{207 \times 10^6 \times 10^{-4}}\ mm$$

$= -0.644\ mm$ (Upward deflection) Ans.

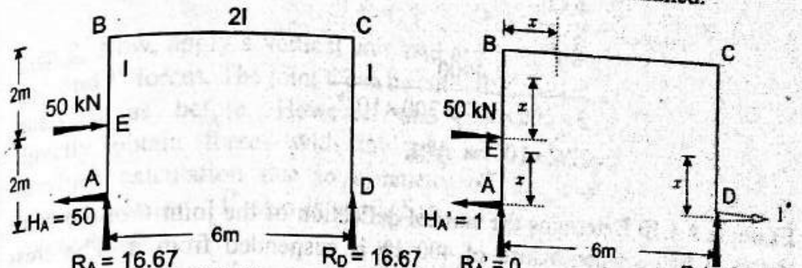
Example # 4.9 Determine horizontal displacement of joint D under the application of 50 kN load at E in the frame. Relative I values are indicated along the members. Take $E = 200 \times 10^6\ kN/m^2$ and $I = 300 \times 10^{-6}\ m^4$



a) Given frame

Fig. 4.12

Sol^n. The problem is solved as similar to the problem of beam. To find the expressions for M and M^*, the support reactions are first determined.



a) For M

b) For M^*

First let us find the reactions for Fig. (a) and (b) respectively

$\sum F_x = 0, \quad H_A = 50\ kN$

$\sum M_A = 0, \quad R_D \times 6 - 50 \times 2 = 0$

or, $R_D = \frac{100}{6} = \frac{50}{3} = 16.67\ kN$

$\sum F_y = 0, \quad -R_A + R_D = 0,$ or, $R_A = R_D = 16.67$

For Fig. (b)

$\sum F_x = 0, \quad H_A' = 1$

$\sum M_A = 0, \quad R_D' \times 6 = 0,\ R_D' = 0$

$\sum F_y = 0, \quad R_A' + R_D' = 0,\ R_A' = 0$

Now, writing the expressions for M and M^* in tabular form,

Section	Orig in	Limits for x	M_x	M^*	$MM^*\frac{dx}{EI}$
AE	A	$0 \to 2m$	$-50x$	$-x$	$\frac{1}{EI}\int_0^2 50x^2 dx = \frac{400}{3EI}$
EB	E	$0 \to 2m$	$-50(2+x)$ $+50x = -100$	$-(2+x)$	$\frac{1}{EI}\int_0^2 100(2+x)dx = \frac{600}{EI}$
BC	B	$0 \to 6m$	$\frac{50}{3}x - 100$	-4	$\frac{1}{2EI}\int_0^6 4\left(100 - \frac{50}{3}\right)dx = \frac{600}{EI}$
DC	D	$0 \to 4m$	0	$-x$	$0 = 0$
					$\Sigma = \frac{4000}{3EI}$

Note: I value for BC is $2I$ and for all other members, it is I. These need to be substituted correctly as in the 6th column of the above table.

Now,

$$\delta_{D(horizontal)} = \frac{4000}{3EI}$$

$$= \frac{4000}{3\times 200\times 10^{6}\times 300\times 10^{-6}}$$

$$= 22.2\times 10^{-3}\ m\ \underline{\text{Ans.}}$$

Example # 4.10 Determine the vertical deflection of the joint C of the crane shown in figure when a load of 400 kN is suspended from it. The cross sectional areas of AC and BC are 30 and 70 cm^2 respectively. Take $E = 2\times 10^{6}\ kg/cm^{2}$

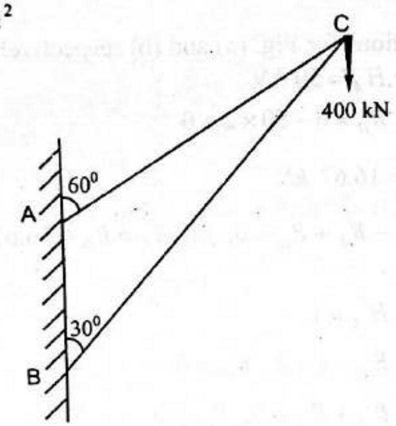
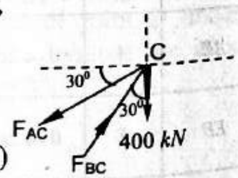


Fig. 4.13

Sol^n.

Step 1. Let us first find the forces F on the members due to the external load by using method of joint. Cutting the joint C,



$\sum F_x = 0$,

or, $F_{AC}\cos 30^0 - F_{BC}\cos 60^0 = 0$

or, $F_{AC} = 0.577\ F_{BC}$ .... (i)

$\sum F_y = 0$,

or, $-F_{AC}\sin 30^0 + F_{BC}\cos 30^0 - 400 = 0$ .... (ii)

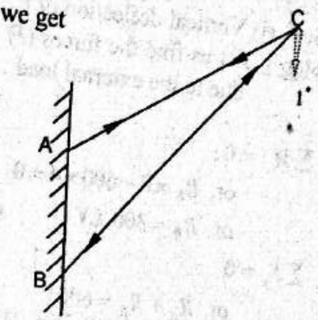
Substituting the value of F_{AC} in equation (ii) we get,

$$-0.577\ F_{BC}\sin 30^0 + F_{BC}\cos 30^0 = 400$$

or, $$F_{BC} = \frac{400}{\left(-0.577\sin 30^0 + \cos 30^0\right)} = 692.61\ kN\ \ \text{(Compression)}$$

Substituting this value of F_{BC} in equation (i), we get

$F_{AC} = 400\ kN$ (Tension)



Step 2. Now, apply a vertical unit load at C and find F^* forces. The joint C can be cut and analysed as before. However, one can directly obtain forces with the help of previous calculation due to symmetry of load application. (1^* also is applied at C as 400 kN load).

$$F_{AC} = \frac{400}{400} = 1\ \text{(Tension)}$$

$$F_{BC} = \frac{692.61}{400} = 1.732\ \text{(Compression)}$$

Step 3. Now, complete the following table using +ve sign for tension forces and –ve sign for compression forces.

Member	ℓ	A	$\frac{\ell}{AE}$	F	F^*	$FF^*\frac{\ell}{AE}$
AC	500	30	$\frac{16.67}{E}$	400	1	$\frac{6668}{E}$
BC	866	70	$\frac{12.37}{E}$	692.61	1.732	$\frac{14839.06}{E}$

$$\delta_v = \sum FF^*\frac{\ell}{AE} = \frac{21507.06}{E}$$

Substituting the value of E, we get

$$\delta_v = \frac{21507.06}{2\times 10^{4}} = 1.075\,cm\ \underline{\text{Ans.}}$$

Example # 4.11 Determine horizontal and vertical deflections of joint C of the truss shown below. Take area of member AB as 10 cm^2 and those of members AC and BC as 15 cm^2. $E = 2\times 10^{5}\ kN/cm^{2}$.

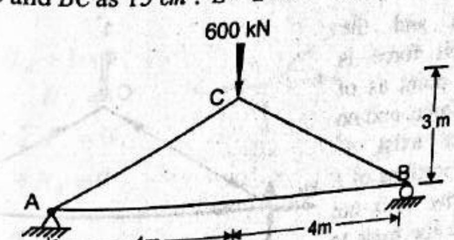
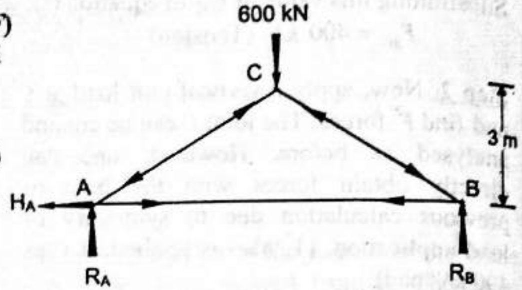


Fig. 4.14

Sol^n. (i) Vertical deflection of C

Step 1. Let us find the forces (F) due to the external load



$\Sigma M_A = 0$,

or, $R_B \times 8 - 600 \times 4 = 0$

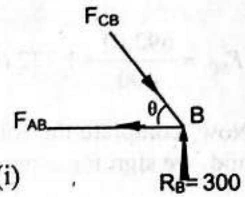
or, $R_B = 300\ kN$

$\Sigma F_y = 0$,

or, $R_A + R_B = 600$

$\therefore\ R_A = 300\ kN$

At joint B



$\theta = \tan^{-1}\dfrac{3}{4} = 36.87 = 36.87$

$\Sigma F_x = 0$

or, $-F_{AB} + F_{CB}\ \cos 36.87^0 = 0$

or, $F_{AB} = F_{CB}\ \cos 36.87^0$ ....(i)

$\Sigma F_y = 0$,

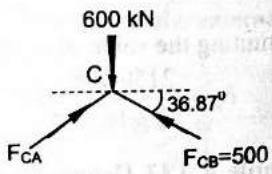
$-F_{CB}\ \sin 36.87^0 + 300 = 0$

or, $F_{CB} = 500\ kN(C)$

Substituting this in equation (i)

$F_{AB} = 400\ kN(T)$

At joint C

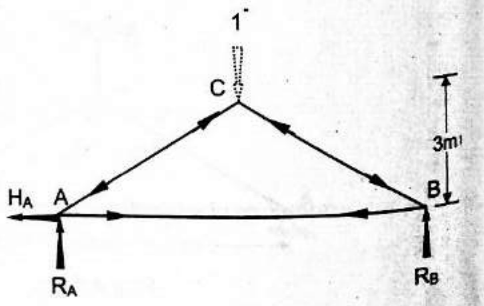


$F_{CA}\cos 36.87^0 - 500\cos 36.87^0 = 0$

or, $F_{CA} = 500\ (C)$

Step 2. Apply the unit vertical force and find F^* forces

Since the location and the direction of the unit force is exactly at the same point as of the external applied force, and no other external force exist, F^* forces will be at proportions of F forces as given below. It is not necessary to analyze the truss to



find the member forces due to F^* in such cases.

$$F_{AB} = \frac{400}{600} = 0.67\ kN$$

$$F_{CB} = \frac{500}{600} = 0.67\ kN$$

$$F_{AC} = \frac{500}{600} = 0.83\ kN$$

Step 3. Tabulation of the forces and calculation of deflection, $\left(E = 2 \times 10^5\ kN/cm^2\right)$

Member	ℓ	A	$\dfrac{\ell}{AE}$	F	F^*	$FF^*\dfrac{\ell}{AE}$
AC	500	15	1.67×10^{-4}	−500	−0.67	0.056
BC	500	15	1.67×10^{-4}	−500	−0.67	1.056
AB	800	10	4×10^{-4}	+400	+0.83	0.133

$$\delta_v = \frac{\Sigma FF^* \ell}{AE} = 1.245\,cm$$

Ans.

Horizontal Deflection of C

Step 1. Since joint B in a roller and the joint C is pin jointed, the point will move in horizontal direction towards right. The joint A being hinged, it cannot move. As we need to find the horizontal deflection, unit horizontal force will be applied at the point.

Step 2. The force in the members due to external load will be same as before. Now, apply the horizontal unit force at C in the direction shown and find the forces. (F^*)

$\Sigma M_A = 0$,

$R_B \times 8 - 1 \times 3 = 0$

or, $R_B = \dfrac{3}{8}$

$\Sigma F_x = 0$,

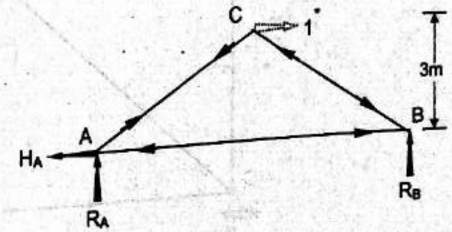
$-H_A + 1 = 0$,

$H_A = 1$

$\Sigma F_y = 0$,

$R_A + R_B = 0$

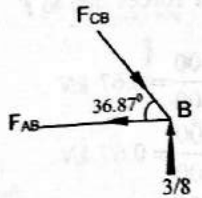
$\therefore\ R_A = -\dfrac{3}{8}$



<u>At joint B</u>

$$F_{CB}\sin 36.87^0 = \frac{3}{8}$$

or, $F_{CB} = 0.625\ (C)$

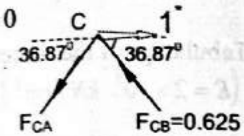


$$F_{AB} = F_{CB}\cos 36.87^0 = 0.5\ (T)$$

<u>At joint C</u>

$$-F_{AC}\cos 36.87^0 - F_{CB}\cos 36.87^0 + 1 = 0$$

or, $F_{AC} = 0.625(T)$



<u>Step 3</u>. Tabulation of the forces and calculation of deflection
$E = 2\times 10^5\ kN/m^2$

Member	ℓ	A	$\frac{\ell}{AE}$	F	F^*	$FF^*\frac{\ell}{AE}$
AC	500	15	1.67×10^{-4}	-500	+0.625	-0.052
BC	500	15	1.67×10^{-4}	-500	-0.625	+0.052
AB	500	10	4×10^{-4}	+400	+0.5	+0.08

$$\delta_{CH} = \frac{\sum FF^*\ell}{AE} = 0.08\ cm\ \underline{\text{Ans.}}$$

The calculation of vertical and horizontal deflections can also be made in the same table. The next example illustrates it.

Example # 4.12 Calculate the horizontal and vertical deflections of the joint B of the truss shown below.

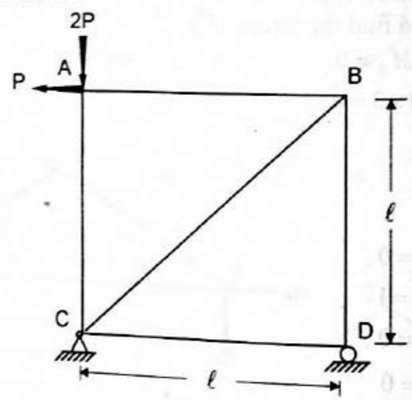
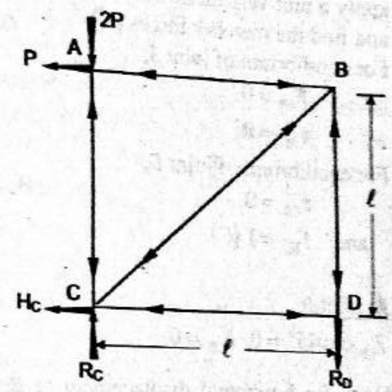


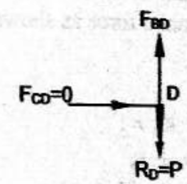
Fig. 4.15

Sol^n.
Step1. **Let us first find the forces F due to the external forces.**

$\sum M_c = 0,\quad R_D\times\ell = P\times\ell,$
$\qquad R_D = P$
$\sum F_x = 0\quad H_c = P$
$\sum F_y = 0,\quad R_c - R_D - 2P = 0$
$\qquad$ or, $R_c = 2P + R_D$
$\qquad$ or, $R_c = 3P$

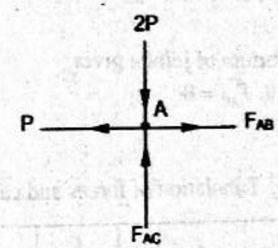


<u>At Joint D</u>
$\sum F_x = 0,\ F_{CD} = 0,$
$\sum F_y = 0,\ F_{BD} = P\ (T)$



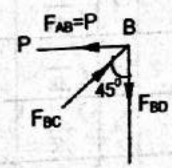
<u>At Joint A</u>

$\sum F_x = 0,\quad F_{AB} = P\ (T)$
$\sum F_y = 0,\quad F_{AC} = 2P\ (C)$



<u>At Joint B</u>
$\sum F_x = 0,\ F_{BA} = F_{BC}\cos 45^0$

or, $F_{BC} = \dfrac{P}{\cos 45^0} = 1.41\ (C)$



Now, for vertical displacement at B, apply a unit vertical downward load at B and find the member forces (F^*)

For equilibrium of joint A,

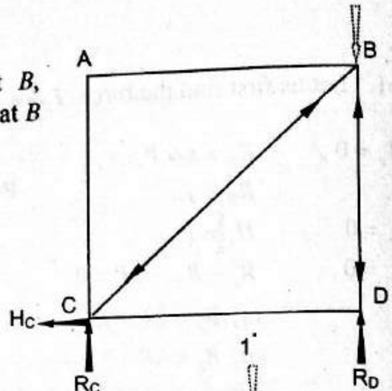
$$F_{AB} = 0$$

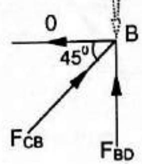
$$F_{AC} = 0$$

For equilibrium of joint D,

$$F_{CD} = 0$$

and $F_{BD} = 1\,(C)$





At joint B

$F_{CB}\cos 45^0 = 0,\ F_{CB} = 0$

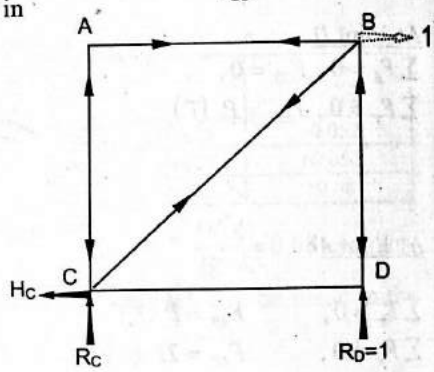
Now, for horizontal displacement at B, apply a unit horizontal force as shown in the figure.

$\Sigma M_c = 0$,

$R_D \times \ell = 1 \times \ell \quad \therefore R_D = 1$

$\Sigma F_x = 0$ at joint D gives

$F_{CD} = 0,\ \Sigma F_y = 0,\ F_{BD} = 1\,(C)$



Equilibrium of joint A gives

$F_{AB} = 0,\ F_{AC} = 0$

Step 3. Tabulation of forces and calculation of deflection

Member	ℓ	A	$\frac{\ell}{AE}$	F	F^*	$FF_V^*\frac{\ell}{AE}$	F_h^*	$FF_h^*\frac{\ell}{AE}$
AB	ℓ	A	$\frac{\ell}{AE}$	P	0	0	0	0
AC	ℓ	A	$\frac{\ell}{AE}$	$-2P$	0	0	0	0
BD	ℓ	A	$\frac{\ell}{AE}$	P	-1	$-\frac{P\ell}{AE}$	-1	$\frac{P\ell}{AE}$
CD	ℓ	A	$\frac{\ell}{AE}$	0	0	0	0	0
CB	$1.41\,\ell$	A	$\frac{1.41\ell}{AE}$	$-1.41P$	1.41	$-\frac{2.803P\ell}{AE}$	0	0

$$\delta_v = \Sigma\frac{FF_V^*\ell}{AE} = -\frac{3.802P\ell}{AE}\ \underline{\text{Ans.}},\ \delta_H = \frac{FF_h^*\ell}{AE} = \frac{P\ell}{AE}\ \underline{\text{Ans.}}$$

The negative δv indicates that the deflection of the joint is opposite to the direction of unit vertical force applied.

Example # 4.13 **Find the vertical deflection of the joint E of the truss shown below. Take E and A as constant for all the members. The lengths of all the members are equal.**

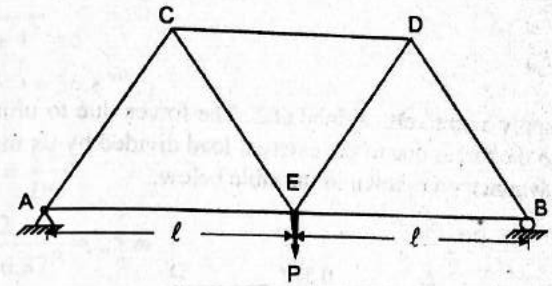


Fig. 4.16

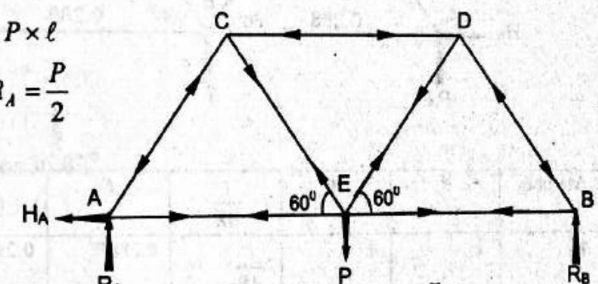
Sol^n.

Step 1 First determine the forces due to the external force.

$\Sigma M_A = 0$,

or, $\quad R_B \times 2\ell = P \times \ell$

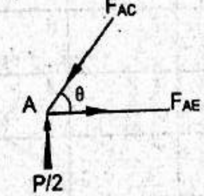
or, $\quad R_B = \frac{P}{2},\ R_A = \frac{P}{2}$



At joint A,

$\Sigma F_x = 0,\ F_{AE} = F_{AC}\cos 60^0 = 0.5\,F_{AC}$

$\Sigma F_y = 0,\ F_{AC}\sin 60^0 = \frac{P}{2},\ F_{AC} = 0.577P$

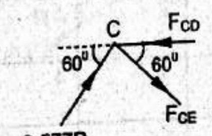


Substitution of value of F_{AC} above gives,

$$F_{AE} = 0.288P$$

At joint C,

$\Sigma F_x = 0$,



or, $0.577P\cos 60^0 + F_{CE}\cos 60^0 = F_{CD}$

$\Sigma F_y = 0, \quad$ or, $0.577P\sin 60^0 = F_{CE}\sin 60^0$

or, $F_{CE} = 0.577P$

Substituting F_{AC} above,

$F_{AC} = 0.577P\cos 60^0 + 0.577P\cos 60^0 = 0.577P$

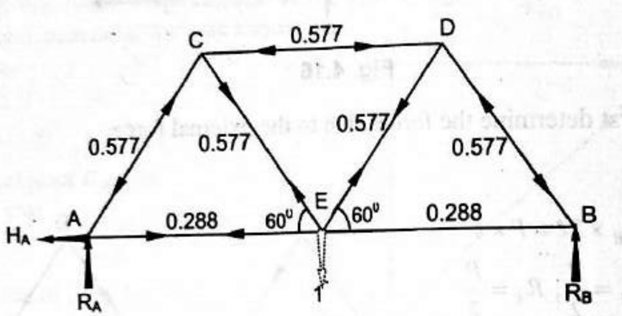
By symmetry of loading,

$F_{AC} = F_{DB}$

$F_{CE} = F_{DE}$

$F_{AE} = F_{EB}$

Step 2. Now, apply a unit vertical load at E. The forces due to unit loads are equal to the forces due to the external load divided by its magnitude P due to symmetry as shown in the table below.

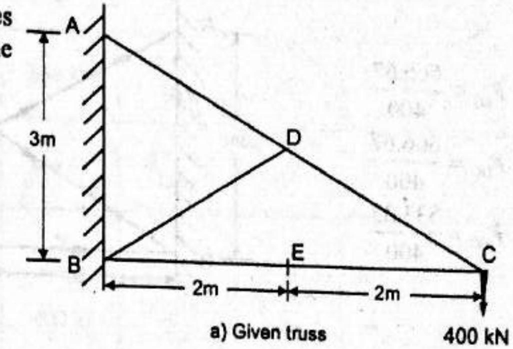


Member	ℓ	A	$\frac{\ell}{AE}$	F	F_v^*	$FF^*\frac{\ell}{AE}$
AE	ℓ	A	$\frac{\ell}{AE}$	$0.288P$	0.288	0.08
AC	ℓ	A	$\frac{\ell}{AE}$	$-0.577P$	-0.577	00.33
CD	ℓ	A	$\frac{\ell}{AE}$	$-0.577P$	-0.577	0.33
CE	ℓ	A	$\frac{\ell}{AE}$	$0.577P$	0.577	0.33
EB	ℓ	A	$\frac{\ell}{AE}$	$0.288P$	0.288	0.08
DB	ℓ	A	$\frac{\ell}{AE}$	$-0.577P$	-0.577	0.33
DE	ℓ	A	$\frac{\ell}{AE}$	$-0.577P$	-0.577	0.33

$$\delta_v = \frac{\Sigma FF^*\ell}{AE} = \frac{1.825P\ell}{AE} \quad \underline{\text{Ans.}}$$

Example # 4.14 Find the vertical and horizontal deflection at node *C* for the truss shown. Take *A* and *E* constant for all the members.

Step 1. First find the forces in the members (F) due to the external load.



a) Given truss

Fig. 4.17

Sol^n.

$AC = \sqrt{3^2 + 4^2} = 5$

$\angle C = \tan^{-1}\frac{3}{4} = 36.87^0$

$\cos 36.87^0 = \frac{EC}{DC}$

$DC = \frac{2}{\cos 36.87^0} = 2.5\ m$

$= AD,$

$DE = \sqrt{2.5^2 - 2^2} = 1.5\ m$

$BD = \sqrt{1.5^2 + 2^2} = 2.5\ m$

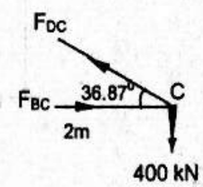
At joint C,

$F_{DC}\sin 36.87^0 = 400$

or, $F_{DC} = 666.67(T)$

$F_{BC} = F_{DC}\cos 36.87^0$

$= 533.33\ (C)$

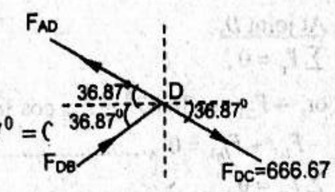


At joint D,

$\Sigma F_x = 0,$

$-F_{AD}\cos 36.87^0 + F_{DB}\cos 36.87^0 + 666.67\cos 36.87^0 = 0$

or, $F_{AD} - F_{DB} = 666.67$ .... (i)



$\Sigma F_y = 0,$

$F_{DB}\sin 36.87^0 + F_{AD}\sin 36.87^0 - F_{DC}\sin 36.87^0 = 0$

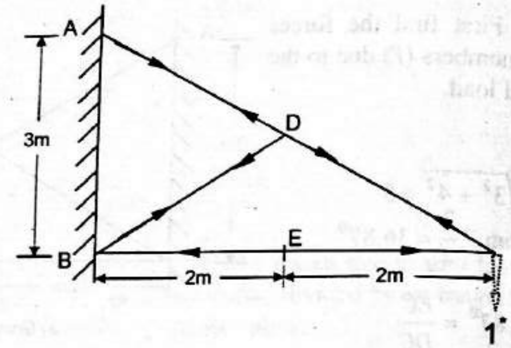
$F_{AD} + F_{DB} = 666.67$ .... (ii)

$F_{AD} - F_{DB} = 666.67$ .... (i)

Solving, $F_{AD} = 666.67(T), \quad F_{DB} = 0$

Step 2. Apply a unit vertical downward force at C and find the forces $\left(F^{\bullet}\right)$.

$$F_{AD}=\frac{666.67}{400}$$
$$F_{DC}=\frac{666.67}{400}$$
$$F_{BC}=\frac{533.33}{400}$$



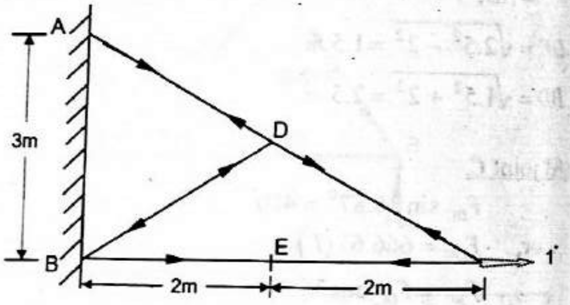
Now, apply a horizontal unit force at C and find the forces $\left(F^{\bullet}\right)$ in the members.

$\sum F_y=0$,

$F_{DC}\sin 36.87^0=0$

or, $\quad F_{DC}=0$

$\quad F_{BC}=1$

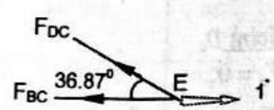


At joint C,

$\sum F_y=0,\ F_{DC}\sin 36.87=0$

or, $F_{DC}=0$

$\therefore\ F_{BC}=1\,kN$



At joint D,

$\sum F_x=0$,

or, $-F_{DA}\cos 36.87^0+F_{DB}\cos 36.87^0=0$

$-F_{DA}+F_{DB}=0$ .... (i)

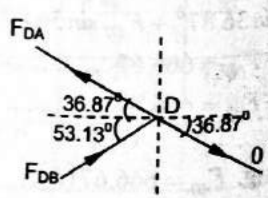
$\sum F_y=0$,

$F_{DA}\sin 36.87^0+F_{DB}\sin 36.87^0=0$

or, $F_{DA}+F_{DB}=0$ .... (ii)

Solving (i) and (ii) we get,

$F_{DA}=0,\qquad F_{DB}=0$



Step 3. Tabulation of forces and calculation of deflection.

Member	ℓ	A	$\frac{\ell}{AE}$	F	$F_v^{\bullet}$	$FF_v^{\bullet}\frac{\ell}{AE}$	$F_h^{\bullet}$	$FF_h^{\bullet}\frac{\ell}{AE}$
AD	2.5	A	$\frac{2.5}{AE}$	666.67	$\frac{666.67}{400}$	$\frac{2777.80}{AE}$	0	0
DC	2.5	A	$\frac{2.5}{AE}$	666.67	$\frac{666.67}{400}$	$\frac{2777.80}{AE}$	0	0
BD	2.5	A	$\frac{2.5}{AE}$	0	0	0	0	0
BC	2.5	A	$\frac{4}{AE}$	−533.33	$\frac{-533.33}{400}$	$\frac{2844.40}{AE}$	1	$\frac{-21.33}{AE}$

$\delta_v=\sum FF_v^{\bullet}\frac{\ell}{EA}=\frac{8400}{EI}$ Ans. $\quad \delta_h=\sum FF_h^{\bullet}\frac{\ell}{AE}=\frac{-21.33}{AE}$ Ans.

-ve sign shows that the deflection is opposite to the direction of unit horizontal force applied.

Example # 4.15 **Find the horizontal displacement of joint D of the pin jointed framed structure shown. Each bar has a cross sectional area A and Young modulus of elasticity E.**

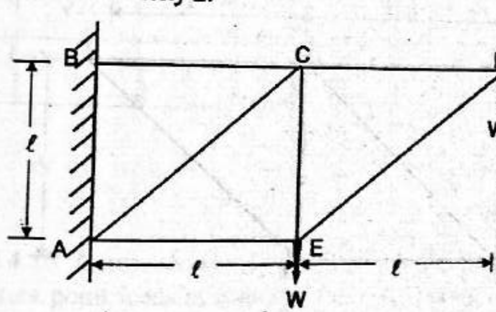


Fig. 4.18

Sol^n.

Step 1. **Force in members (F) due to external load.**

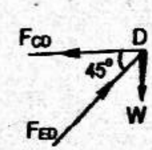
At joint D,

$F_{ED}\sin 45^0=W$

or, $\quad F_{ED}=1.41W\ (C)$

$\quad F_{ED}\cos 45^0=F_{CD}$

or, $\quad F_{CD}=1.41\ W\cos 45^0=W\ (T)$

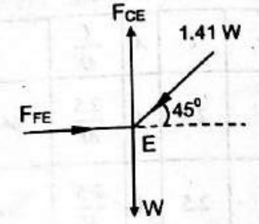


At joint E,

$$1.41W\cos 45^0 = F_{FE}$$

or, $F_{FE} = W(C)$

or, $F_{CE} = W + 1.41W \sin 45^0 = 2W(T)$



At joint C,

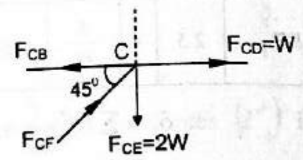
$$W - F_{CB} + F_{CA}\cos 45^0 = 0 \text{.....................(i)}$$

or, $F_{CA}\sin 45^0 = 2W$

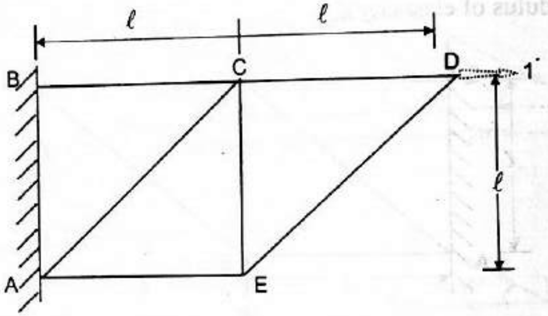
or, $F_{CA} = \dfrac{2W}{\sin 45^0} = 2.83W(C)$

Substituting this in equation (i)

$$F_{CB} = W + 2.83W\cos 45^0 = 3W(T)$$



Step 2. Now Apply a unit horizontal force at D and calculate the member forces (F^*)

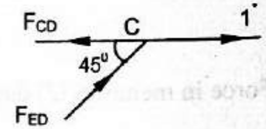


At joint D,

or, $-F_{CD} + F_{ED}\cos 45^0 + 1 = 0$

and $F_{ED}\sin 45^0 = 0$, or $F_{ED} = 0$

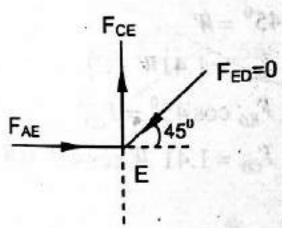
$\therefore$ $F_{CD} = 1(T)$



At joint E,

$F_{FE} = 0$

$F_{CF} = 0$

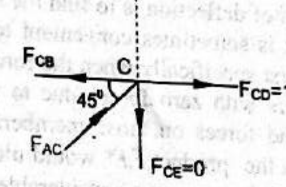


At joint C,

$$F_{AC}\sin 45^0 = 0$$

or, $F_{AC} = 0$

$F_{CB} = 1(T)$



Step 3. Tabulation of forces and calculation of deflection.

Members	ℓ	A	$\frac{\ell}{AE}$	F	F^*	$FF^*\frac{\ell}{AE}$
BC	ℓ	A	$\frac{\ell}{AE}$	$3W$	1	$\frac{3W\ell}{AE}$
CD	ℓ	A	$\frac{\ell}{AE}$	W	1	$\frac{W\ell}{AE}$
DE	$\sqrt{2}\ell$	A	$\frac{\sqrt{2}\ell}{AE}$	$1.41W$	0	0
AE	ℓ	A	$\frac{\ell}{AE}$	$-W$	0	0
AC	$\sqrt{2}\ell$	A	$\frac{\sqrt{2}\ell}{AE}$	$-2.83W$	0	0
CE	ℓ	A	$\frac{\ell}{AE}$	$2W$	0	0

$$\delta_H = \frac{\sum FF^* \ell}{AE} = \frac{4W\ell}{AE} \quad \underline{\text{Ans.}}$$

Example # 4.16 A truss shown in Fig. (4.20) is hinged at A and roller support at D. It carries point loads at B and C. Find the horizontal deflection of the support D, if the area of cress section of each member is 50 cm^2. Take $E = 2\times10^6\, kg/cm^2$

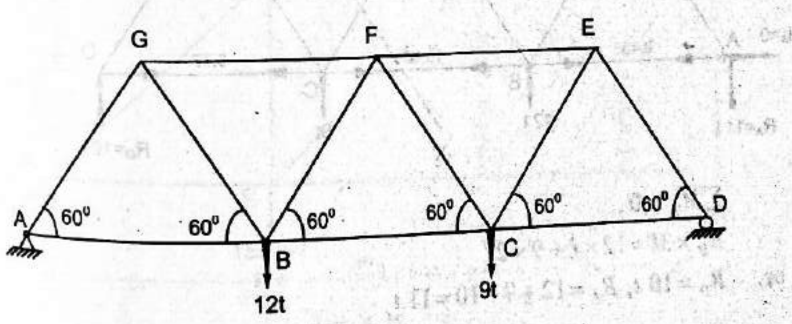
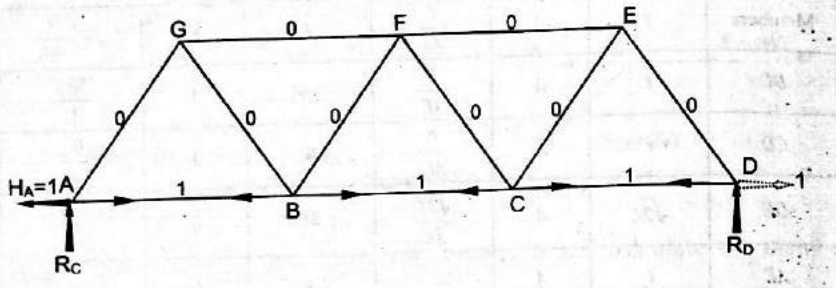


Fig. 4.20

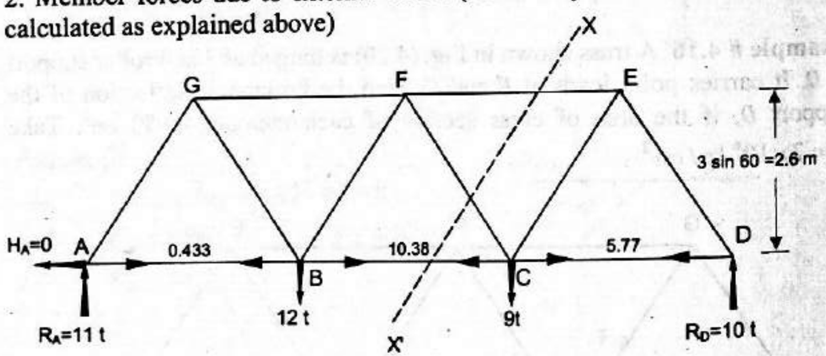
Sol^n^. It is clear from the previous examples that the first step in the computation of deflection is to find the member forces due to external forces. However, it is sometimes convenient to find the member forces due to the unit force first specifically when the forces in a number of members are zero. The members with zero forces due to the application of unit force is then identified and forces on those members due to the external forces are not computed as the product $F.F^*$ would ultimately turn into a zero quantity (F^* being zero). This can save a considerable amount of computational work. The following example illustrates this procedure.

1. Forces due to unit horizontal force applied at D.



On careful observation of the above figure, it is clear that the unit force induces member forces only on the bottom members. The forces in all the other members are zero. Only a mental calculation is required to know this. For example at joint D, since $R_D = 0$, vertical component of ED has to be zero. We can proceed to other joints from D to determine the forces in other members.

2. Member forces due to external loads. (Note: only bottom member forces calculated as explained above)



$$\sum M_A = 0,$$

$$R_D \times 3\ell = 12 \times \ell + 9 \times 2\ell$$

or, $\quad R_D = 10\ t,\ R_A = 12 + 9 - 10 = 11\ t$

Joint D,

$$F_{ED} \sin 60^0 = 10, \quad F_{ED} = 11.55\ t\ (C)$$

$$F_{ED} \cos 60^0 = F_{CD} = 5.77\ t\ (C)$$

Joint A,

$$F_{AG} \sin 60^0 = R_A = 11 \quad \text{or, } F_{AG} = 12.7\ (C)$$

$$F_{AG} \cos 60^0 = F_{AB}$$

or, $\quad F_{AB} = 6.35$

To find the force in BC, pass the section XX' and take moment about F considering equilibrium of left hand portion of the truss.

$$F_{BC} \times 2.6 + 12 \times \frac{3}{2} - 11 \times 4.5 = 0$$

or, $\quad F_{BC} = 12.11(T)$

3. Tabulate the forces and calculate the deflection. Take $E = 2 \times 10^3 t/cm^2$

Members	ℓ	A	$\dfrac{\ell}{AE}$	F	F_h^*	$FF_h^* \dfrac{\ell}{AE}$
AB	300	50	2×10^{-3}	6.35	1	0.019
BC	300	50	2×10^{-3}	12.11	1	0.036
CD	300	50	2×10^{-3}	5.77	1	0.017

$\Sigma = 0.072$

$$\therefore \delta_{DH} = \frac{\sum FF^* \ell}{AE} = 0.072\ cm \ \underline{\text{Ans.}}$$

Example # 4.17 A steel truss of span 15 m is loaded as shown in Fig.(4.21) The cross sectional area of each member is such that it is subjected to a stress of 100 N/mm^2. Find the vertical deflection of the joint C. Take $E = 200kN/mm^2$.

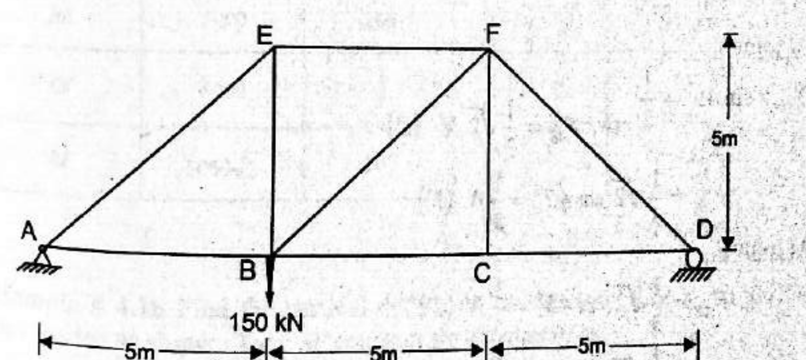
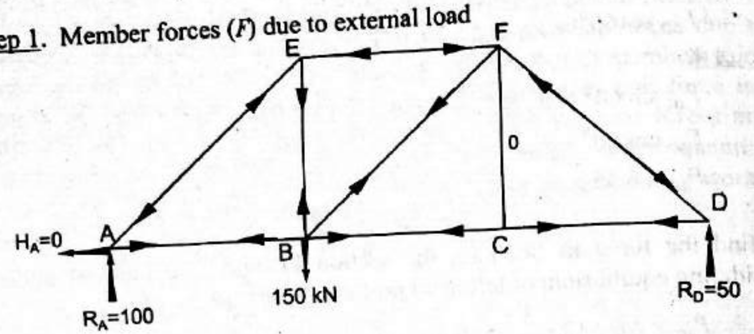


Fig. 4.21

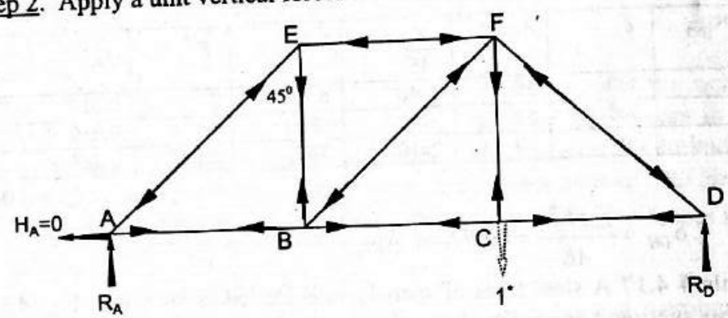
Sol^n.

Step 1. Member forces (F) due to external load



The stresses are given as $\sigma = \frac{F}{A} = 100\ N/mm^2$. Observing the structure carefully, one can produce a force diagram as in the above figure.

Step 2. Apply a unit vertical forces at C and find member forces (F^*)



$$\sum M_A = 0, \qquad R_D \times 15 = 1 \times 10, \qquad R_D = \frac{2}{3}\ N$$

$$R_A = \frac{1}{3}\ N$$

At joint A,

$$F_{AE} \times \sin 45^0 = \frac{1}{3} \quad \text{or,} \quad F_{AE} = \frac{1}{3}\sqrt{2}\ N \quad (C)$$

$$F_{AB} = \frac{1}{3}\sqrt{2}.\cos 45^0 = \frac{1}{3} N\ (T)$$

At joint E,

$$F_{EB} = \frac{1}{3}\sqrt{2}.\cos 45^0 = \frac{1}{3} N\ (T)$$

$$F_{EF} = \frac{1}{3}\sqrt{2}.\sin 45^0 = \frac{1}{3} N\ (C)$$

At joint D,

$$F_{DF} \sin 45^0 = \frac{2}{3}, \quad \text{or } F_{DF} = \frac{2}{3}\sqrt{2}\ (C)$$

$$F_{DC} = \frac{2}{3}\sqrt{2}\cos 45^0 = \frac{2}{3} N\ (T)$$

At joint C,

$$F_{CF} = 1N\ (T)$$

$$F_{CB} = \frac{2}{3} N\ (T) \qquad F_{CB} = F_{DC}\ \frac{2}{3} N\ (T)$$

At joint B,

$$F_{BF} \cos 45^0 = F_{EB} = \frac{1}{3} \quad \text{or,} \quad F_{BF} = \frac{1}{3}\sqrt{2}\ N(C)$$

Step 3. Tabulation of forces and calculation of deflection
$E = 200\ kN/mm^2 = 200000\ N/mm^2$

Members	ℓ	$\frac{F}{A}$	F^*	$FF^* \frac{\ell}{AE}$
AE	$5000\sqrt{2}$	-100	$-\frac{1}{3}\sqrt{2}$	1.67
EF	5000	-100	$-\frac{1}{3}$	0.83
FD	$5000\sqrt{2}$	-100	$-\frac{2}{3}\sqrt{2}$	3.33
DC	5000	100	$\frac{2}{3}$	1.67
CB	5000	100	$\frac{2}{3}$	1.67
BA	5000	100	$\frac{1}{3}$	0.83
BE	5000	100	$\frac{1}{3}$	0.83
CF	5000	0	1	0
BF	$5000\sqrt{2}$	100	$-\frac{1}{3}\sqrt{2}$	-1.67

$$\delta_{cv} = \sum \frac{FF^* \ell}{AE} = 9.16\ mm\ \underline{\text{Ans.}}$$

Example # 4.18 Find the vertical deflection of the node E of a pin jointed truss loaded as shown. Take AE constant for all members.

Sol^n.

Step 1. Member forces due to external load.

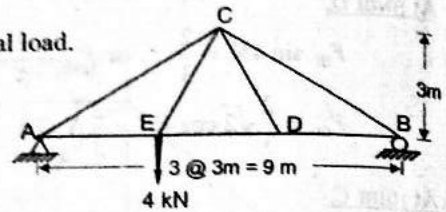


Fig. 4.22

$M_A = 0$

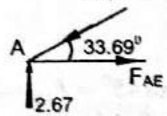
or, $R_B \times 9 = 4 \times 3$

$\therefore \quad R_B = \dfrac{4}{3} = 1.33\ kN$

Again, $R_A + R_B = 4$

$\therefore \quad R_A = 2.667\ kN$

Joint A



$\therefore \quad F_{CA} = 4.808\ kN \quad (C)$

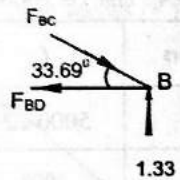
Again, $F_{AE} = F_{AC} \cos 33.69^0$

(since $\angle CAE = \angle CBD = \tan^{-1} \dfrac{3}{4.5} = 33.69^0$)

$= 4.808 \times \cos 33.69^0$

$= 4\ kN \quad (T)$

Joint B



$F_{BC} \sin 33.69^0 = 1.33$

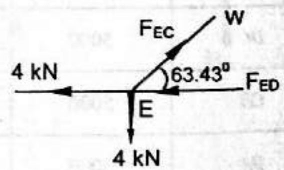
$\therefore \quad F_{BC} = 2.403\ kN\ (C)$

Again $F_{BD} = F_{BC} \cos 33.69^0$

$= 2.403 \times \cos 33.69^0$

$= 2\ kN\ (T)$

Joint E



$F_{EC} \sin 63.43^0 = 4$

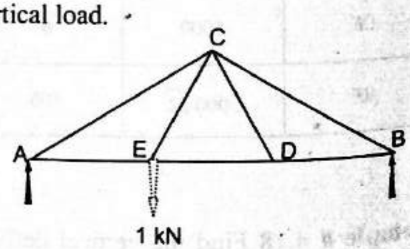
$\therefore \quad F_{EC} = 4.47\ kN\ (T)$

Again $4 - F_{ED} + F_{EC} \times \cos 63.43^0 = 0$

or, $F_{ED} = 2\ kN \quad (C)$

Obviously, we have, $F_{CD} = 0\ kN$

Step 2. Member forces due to unit vertical load.



$M_A = 0$

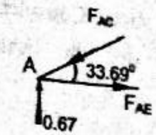
$\therefore \quad R_B \times 9 = 1 \times 3$

$\therefore \quad R_B = \dfrac{1}{3} = 0.33\ kN$

Again, $R_A + R_B = 1$

$\therefore \quad R_A = 0.667\ kN$

Joint A



$F_{AC} \sin 33.69^0 = 0.667$

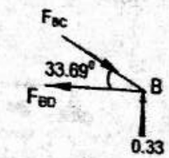
$\therefore \quad F_{AC} = 1.202\ kN \quad (C)$

Again, $F_{AE} = F_{AC} \cos 33.69^0$

$= 1.202 \times \cos 33.69^0$

$= 1\ kN \quad (T)$

Joint B



$F_{BC} \sin 33.69^0 = 0.33$

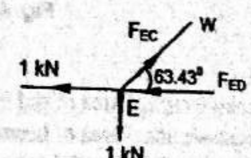
$\therefore \quad F_{BC} = 0.6\ kN\ (C)$

Again $F_{BD} = F_{BC} \cos 33.69^0$

$= 0.6 \times \cos 33.69^0$

$= 0.5\ kN\ (T)$

Joint E



$F_{EC} \sin 63.43^0 = 1$

$\therefore \quad F_{EC} = 1.118\ kN\ (T)$

Again $1 - F_{ED} + F_{EC} \times \cos 63.43^0 = 0$

$F_{ED} = -1.118 \times \cos 63.43^0 + 1$

$= -0.5\ kN\ (C)$

Obviously, we have $F_{CD} = 0\ kN$

Members	ℓ	A	$\dfrac{\ell}{AE}$	F	F'	$FF'\dfrac{\ell}{AE}$
AC	5.408	A	$\dfrac{5.408}{AE}$	-4.808	-1.202	$31.254\dfrac{1}{AE}$
EC	3.35	A	$\dfrac{3.35}{AE}$	4.47	1.118	$16.74\dfrac{1}{AE}$
DC	3.35	A	$\dfrac{3.35}{AE}$	0	0	0
BC	5.408	A	$\dfrac{5.408}{AE}$	-2.403	-0.6	$7.8\dfrac{1}{AE}$
AE	3	A	$\dfrac{3}{AE}$	4	1	$12\dfrac{1}{AE}$
ED	3	A	$\dfrac{3}{AE}$	-2	-0.5	$3\dfrac{1}{AE}$
DB	3	A	$\dfrac{3}{AE}$	2	0.5	$3\dfrac{1}{AE}$

Total deflection $= \sum 73.79 \dfrac{1}{AE}$

Example # 4.19 For the planar structural system shown in Fig. (2.23), made of aluminum, determine the vertical deflection at D due to both bending and direct (axial) stress. Consider only effects of applied load 48 kN. For the rod $A = 500\ mm^2$ for the beam $A = 400\ mm^2$ and $I = 20\times10^6\ mm^4$ and $E = 70\ GPa$.

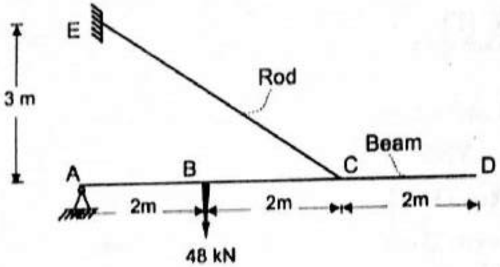


Fig. 4.23

Sol^n.

Given that

Cross-sectional area of rod = 500 mm^2
Cross-sectional area of beam = 4000 mm^2
Moment of inertia of beam = $20\times10^6\ mm^2$
Modulus of elasticity E = 70 GPa

For real loading,

$\sum M_C = 0$

$R_A \times 4 = 48\times2$

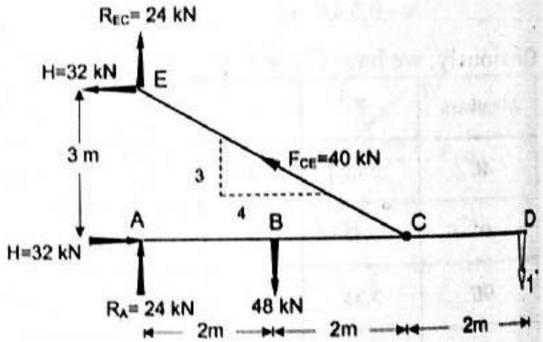
or, $R_A = 24\ kN$

Again, $\sum M_C = 0$

$R_{CE}\times4 = 48\times2$

$\therefore\quad R_{CE} = 24\ kN$

$= R_{EC}$

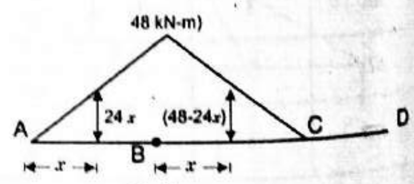


Now, using force triangle for member EC

$$\tan\theta = \frac{3}{4}$$

or $\quad \theta = \tan^{-1}\left(\frac{3}{4}\right) = 36.87^0$

$$F_{CE} = \frac{R_{CE}}{\sin\theta} = \frac{24}{\sin 36.87} = 40\ kN$$



M-diagram

Again,

$$H_{CE} = \sqrt{40^2 - 24^2}$$
$$= 32\ kN$$

For virtual loading,

When unit load (1 kN) load is acting downward at D,

$\sum M_C = 0$

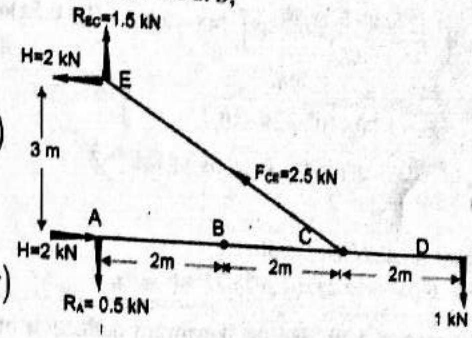
$R_A \times 4 = 1\times2$

$\therefore\quad R_A = \frac{2}{4} = 0.5\ kN(\downarrow)$

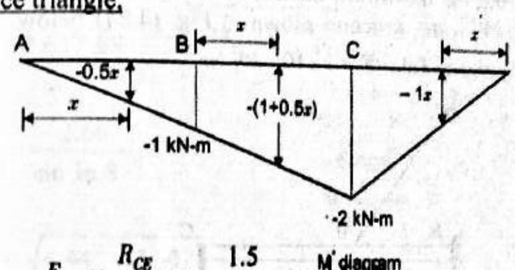
Again, $\sum M_A = 0$

$R_{EC}\times4 = 1\times6$

$R_{EC} = \frac{6}{4} = 1.5\ kN(\uparrow)$



Using force triangle,



$$F_{CE} = \frac{R_{CE}}{\sin\theta} = \frac{1.5}{\sin 36.87^0} = 2.5\ kN$$

Again, $\quad H_{CE} = \sqrt{2.5^2 - 1.5^2}$
$= 2\ kN$

Here, when 1 kN vertical force is applied at C, this force causes are axial force on member CE and in the part AC of the beam. Owing to this force bending moments are also caused in the beam AD. So deflection at point C depends on axial force and flexure.

Hence deflection, $\quad \Delta = \sum\frac{FF^*\ell}{AE} + \int_0^{\ell}\frac{MM^*dx}{EI}$

Member	$F\ (kN)$	$F^*(kN)$	$\ell\ (m)$	$A\ (m^2)$	$FF^*\ell/A$
AC	32	2	4	40×10^{-4}	64000
CE	40	2.5	5	5×10^{-4}	1000000

$$\sum\frac{FF^*\ell}{A} = 1064000$$

or, $\dfrac{\sum FF^{*}\ell}{AE}=\dfrac{1064000}{70\times106}=0.0152=15.2\ mm$

Again, $\displaystyle\int_0^{\ell}\frac{MM^{*}dx}{EI}=\int_{AB}\frac{MM^{*}dx}{EI}+\int_{BC}\frac{MM^{*}dx}{EI}+\int_{CD}\frac{MM^{*}dx}{EI}$

$$=\int_0^2\frac{(24x)(-0.5x)dx}{EI}+\int_0^2(48-24x)-(1+0.5x)dx+\int_0^2 0\times\frac{(-1x)\times dx}{EI}$$

$$=-\frac{96}{EI}=-\frac{96}{70\times10^{6}\times20\times10^{-6}}$$

$$=-\frac{96}{1400}=-0.06857\ m=-68.57\ mm$$

Now, total displacement
$=(-68.57+15.2)\ mm=53.37\ mm$ Ans.

Example # 4.20 Find the downward deflection of the end C caused by the applied force of 5 kN in the structure shown in Fig. (4.24) below. Neglect deflection caused by shear. Take $E=7\times10^{7}\ kN/m^{2}$

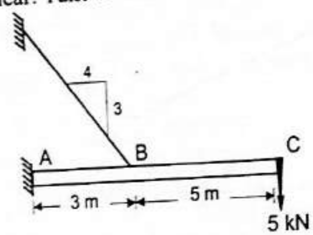
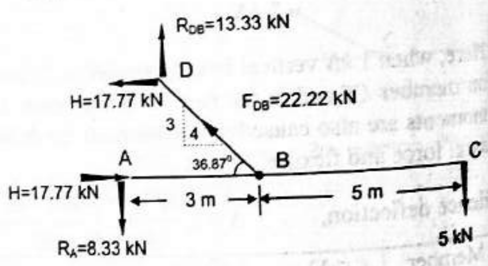


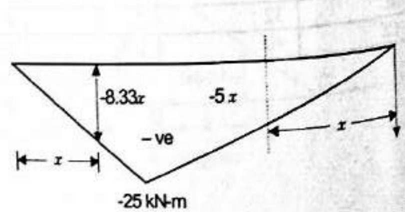
Fig. 4.24

Solⁿ.



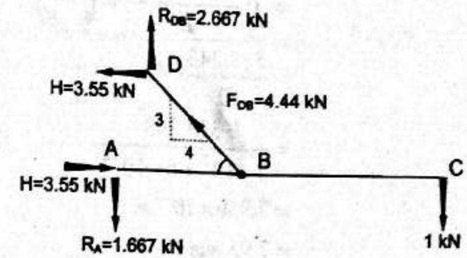
For real loading
$\sum M_B=0$
$5\times5=R_A\times3$
$\therefore R_A=8.33\ kN$
Again, $\sum M_A=0$
$R_{DB}\times3=5\times8$
(R_{DB} = Vertical force in member DB)
$\therefore R_{DB}=5\times\dfrac{8}{3}=13.33\ kN$

Now, $F_{DB}=\dfrac{R_{DB}}{\sin\theta}$ (axial force on DB)



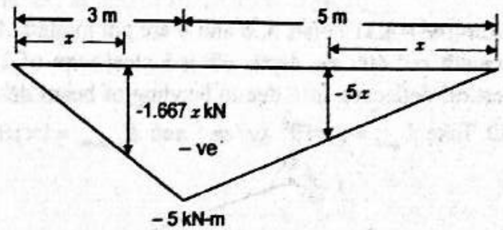
M - diagram

$=\dfrac{13.33}{\sin 36.87^{0}}$
$=22.22\ kN$
$H_{DB}=\sqrt{22.22^{2}-13.33^{2}}$
$=17.77\ kN$



Now, Again for virtual loading
$\sum M_B=0$
$5\times1=R_A\times3$
$\therefore R_A=\dfrac{5}{3}=1.667\ kN$

Again, $\sum M_A=0$
$R_{BD}\times3=1\times8$
$\therefore R_{BD}=\dfrac{8}{3}=2.667\ kN$



M diagram

Now,
$F_{DB}=\dfrac{R_{DB}}{\sin\theta}=\dfrac{2.667}{\sin 36.87^{0}}$
$=4.44\ kN$
Again $H_{DB}=\sqrt{4.44^{2}-2.667^{2}}$
$=3.55\ kN$

Here, when 1 kN virtual force is applied at C, this force causes an axial force on a member DB and in the part AB of the beam. Owing to this force, bending moments are also caused in the beam AC. So, deflection of point C depends on axial force and flexure. Hence

Deflection $\Delta=\sum\dfrac{FF^{*}\ell}{AE}+\displaystyle\int_0^{\ell}\frac{MM^{*}dx}{EI}$

Here,

Member	F (kN)	F^{*} (kN)	ℓ (m)	A (m^2)	$FF^{*}\ell/A$
DB	22.22	4.44	3.75	$5\times10^{-4}\ m^2$	739926
AB	17.77	3.55	3	$50\times10^{-4}\ m^2$	37850.1

From the table $\sum\dfrac{FF^{*}\ell}{A}=777776.1$

or, $\sum\dfrac{FF^{*}\ell}{AE}=0.0111\ m=11.11\ mm$

Now, $\int_0^l \frac{MM^* dx}{EI} = \int_0^3 \frac{(-1.667x)\times(-8.33x)\,dx}{EI} + \int_0^5 \frac{(-x)(-5x)\,dx}{EI}$

$$= \int_0^3 \frac{13.89x^2\,dx}{EI} + \int_0^5 \frac{5x^2\,dx}{EI}$$

$$= \frac{333.343}{EI}$$

$$= \frac{333.343}{7\times10^7\times6\times10^{-4}}$$

$$= 7.936\times10^{-3}\ m$$

$$= 7.93\ mm$$

$\therefore$ Total deflection

$$\Delta = 7.93 + 11.11$$

$$= 19.04\ mm\ \text{Ans.}$$

Example # 4.21 Points *A*, *B* and *D* are pin jointed. *AB* is *RCC* beam of 200 *mm* breadth and 400 *mm* depth. *BD* is a steel wire of 10 mm diameter. Find the vertical deflection at *C* due to bending of beam *AB* and axial tension of wire *BD*. Take $E_{steel} = 2\times10^6\ kg/cm^2$ and $E_{concrete} = 1\times10^5\ kg/cm^2$

[T.U. 2055 Baishakh]

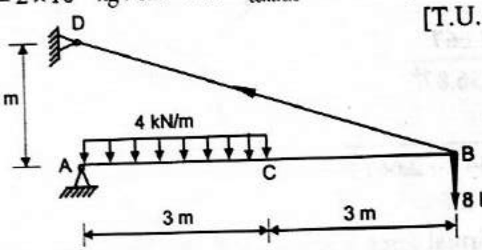


Fig. 4.25

Solⁿ.

For real loading

$$\Sigma M_B = 0$$

$$R_A\times6 = 4\times3\times(3+\frac{3}{2})$$

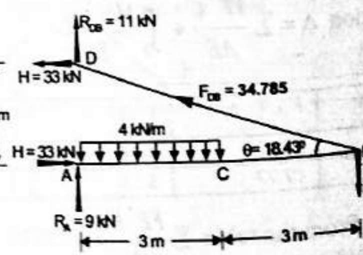
$$\therefore\ R_a = 12\times\frac{4.5}{6}$$

$$= 9\ kN$$

$$\Sigma M_A = 0$$

$$R_{BD}\times6 = 4\times3\times1.5 + 8\times6$$

$$\therefore\ R_{BD} = 11\ kN$$

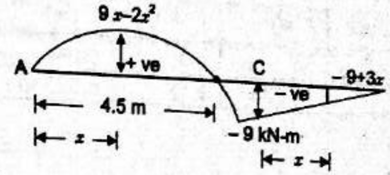


Now using force triangle,

Inclination between wire and beam

$$\theta = \tan^{-1}\left(\frac{2}{6}\right)$$

$$= 18.43^0$$



$$F_{DB} = \frac{R_{DB}}{\sin\theta} = \frac{11}{\sin 18.43^0}$$

$$= 34.785\ kN$$

Horizontal component of F_{DB}

$$= \sqrt{34.785^2 - 11^2}$$

$$= 33\ kN$$

For virtual loading

When 1 *kN* virtual load is acting at point *C* downward direction.

$$\Sigma M_B = 0$$

$$R_A\times6 = 1\times3$$

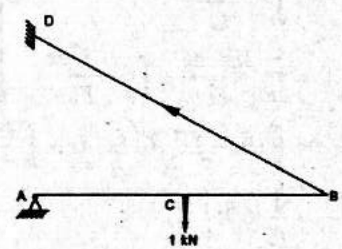
$$\therefore\ R_A = \frac{3}{6} = 0.5\ kN$$

$$\Sigma M_A = 0$$

$$R_{BD}\times6 = 1\times3$$

$$R_{BD} = \frac{3}{6} = 0.5\ kN$$

$$= R_{DB}$$



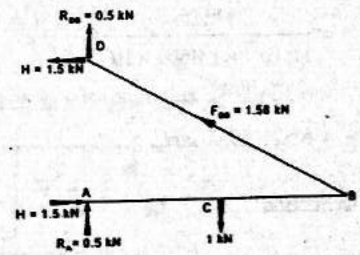
Now, using force triangle

$$F_{DB} = \frac{R_{DB}}{\sin 18.43^0} = 1.58\ kN$$

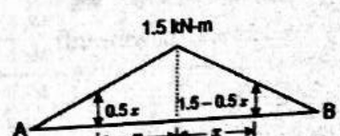
Again, horizontal component of F_{DB}

$$= \sqrt{1.58^2 - 0.5^2}$$

$$= 1.5\ kN$$



When 1 *kN* vertical force is applied at *C*, this force causes axial force on beam *ABD*, owing to this force, bending moments are also caused in the beam *AB*, So, deflection at point *C* depends on axial force and flexure.



Hence, deflection $\Delta = \frac{\Sigma FF^* l}{AE} + \int_0^l \frac{MM^* dx}{EI}$

Now, deflection due to axial force,

Member	F (kN)	F^* (kN)	ℓ (m)	A (m^2)	$FF^*\ell/A$
AB	33	1.5	6 m	0.08	3712.5
BD	34.785	1.58	6.325 m	7.854×10^{-4}	442607.458

Now,

$$\frac{\sum FF^*\ell}{AE}=\frac{FF^*\ell}{AE_S}+\frac{FF^*\ell}{AE_C}$$

$$=\left(\frac{3712.5}{1\times10^{10}}+\frac{442607.46}{2\times10^{8}}\right)\times1000\ mm=2.21\ mm$$

Again, deflection due to flexure for beam AB,

$$=\int_{AB}\frac{MM^*dx}{EI}$$

$$=\int_{AC}\frac{MM^*dx}{EI}+\int_{CB}\frac{MM^*dx}{EI}$$

$$I=\frac{1}{12}bd^3=\frac{1}{12}\times0.2\times0.4^3=1.0667\times10^{-3}\ m^4$$

$$=\int_0^3(9x-2x^2)(0.5x)\,dx+\int_0^3\frac{(-9+3x)(1.5-0.5x)\,dx}{EI}$$

$$=\frac{1}{IE}[6.75]$$

$$=\frac{6.75}{1\times10^{10}\times1.0667\times10^{-3}}$$

$$=6.32\times10^{-7}\ m$$

$$=6.32\times10^{-4}\ mm$$

Total deflection

$$\Delta=\frac{\sum FF^*\ell}{AE}+\int_{AB}\frac{MM^*dx}{EI}$$

$$=2.21+6.32\times10^{-4}\ mm$$

$$=2.21\ mm\ \text{(downward)}$$

Example # 4.22 Determine the vertical deflection of joint E of the truss shown in Fig. (4.26) due to given loading, due to increase of temperature by 20^0 in the member AE and EB and also due to the member CE and DE being 10 mm too long. Given for all members, cross sectional area = 100 mm^2, Young's modulus = 200 kN/mm^2 and $\alpha=12\times10^{-6}/^0C$.

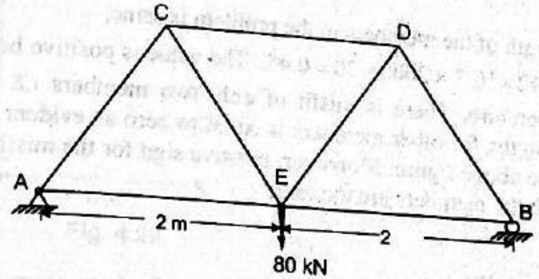


Fig46

Sol^n.

The forces in the truss due to external load, due to temperature change and due to lack of fit are calculated separately. The forces due to external load are calculated similar to the example 4.13 and forces are tabulated directly. Deflection due to temperature change is calculated as

$$\delta=\sum\frac{FF^*\ell}{AE}=\sum F^*\Delta\ell_t=\sum F^*\alpha\ell t$$

The forces and the calculation of defection are tabulated below.

Mem	ℓ	A	$\frac{\ell}{AE}$	F	F^*	$FF^*\frac{\ell}{AE}$	$\Delta\ell_t=\alpha\ell t$	$\Delta\ell_f$	$F^*\Delta\ell$	$F^*\Delta\ell_f$
AE	ℓ	A	0.01	23.04	0.288	0.066	0.48	0	0.138	0
AC	ℓ	A	0.01	−46.16	−0.577	0.266	[illegible]	0	0	0
CD	ℓ	A	0.01	−46.16	−0.577	0.266	[illegible]	0	0	0
CE	ℓ	A	0.01	−46.16	0.577	0.266	[illegible]	10	0	5.77
EB	ℓ	A	0.01	23.04	0.288	0.066	0.48	0	0.138	0
DB	ℓ	A	0.01	−46.16	−0.577	0.266	[illegible]	0	0	0
DE	ℓ	A	0.01	−46.16	−0.577	0.266	[illegible]	10	0	5.77
						$\Sigma=1.46$			$\Sigma=$ 0.276	$\Sigma=$ 11.54

Total deflection = deflection due to external loading + deflection due to temperature change + deflection due to lack of fit.

$$\delta_{total} = \sum \frac{FF^* \ell}{AE} + \sum F^* \Delta \ell_t + \sum F^* \Delta \ell_f$$
$$= 1.45 + 0.276 + 11.54$$
$$= 13.28\ mm$$ Ans.

Discussion

Since the length of the members in the problem is same, $\Delta \ell_t = \alpha \ell t = 12 \times 10^{-6} \times 2000 \times 20 = 0.48$. The value is positive because of the rise of temperature. There is misfit of only two members *CE* and *DE*, the change in lengths for other members is equal to zero as evident from the 9th column of the above figure. Moreover, positive sign for the misfit lengths are taken, as both the members are too long.

4.6 EXERCISES

Ex .1 Find the deflection at the free end of the cantilever beam loaded with triangular load as shown in Fig.(4.27).

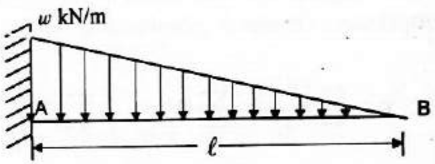


Fig. 4.27

[Ans. $\frac{w\ell^4}{30EI}$]

Ex .2 Find the horizontal deflection of joint *B* and *A* and also find the slope at *C* of the frame in Fig. (4.28)

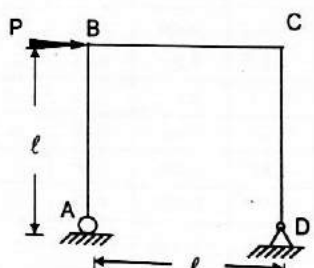


Fig. 4.28

[Ans. $\frac{2P\ell^3}{3EI}, \frac{5P\ell^3}{6EI}, -\frac{P\ell^2}{6EI}$]

Ex .3 Using unit load method determine at *A* and placed on rollers at *D*. Calculate the horizontal movement of *D* and *B* as well as their rotations.

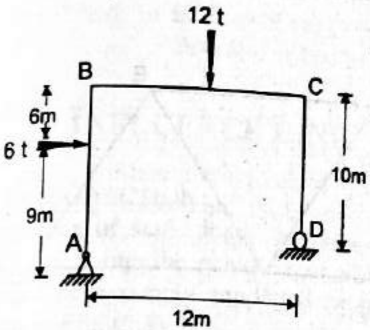


Fig. 4.29

[Ans. $\delta_D = \frac{8857}{EI} m,\ \theta_D = -\frac{108.0}{EI} rad.$
$\delta_B = \frac{7776.0}{EI} m,\ \theta_B = \frac{162.0}{EI} rad.$]

Ex .4 In truss shown in figure, the top chord members have area of 9 cm^2, verticals have area of 8 cm^2 and bottom chord members and diagonals have area of 5 cm^2. Find the particle deflection of joint *I* when load of 20 *kN* each is applied at each joint *G* and *H* as shown in Fig. (4.29). Take $E = 200\ GN/m^2$.

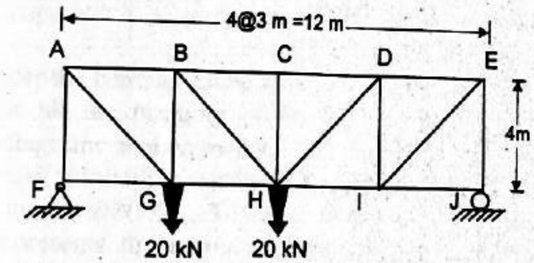


Fig. 4.30

[Ans. 2.325 mm]

Ex .5 In the Fig. (4.30) shown points *A*, *B* and *D* are pin jointed *AB* is a *RCC* beam of size 40 *cm* × 60 *cm* effective *BD* is steel wire of 8 mm diameter. Find the vertical deflection at *C* due to bending of beam *AB* and axial tension of wire *BD*. Take $E_s = 2 \times 10^5\ N/mm^2$ and $E_c = 1 \times 10^4\ N/mm^2$.

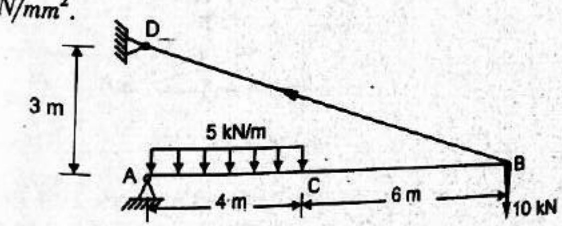


Fig. 4.31

Ex. 6 A pin-jointed frame is hinged at B and supported on a roller at G. Find the vertical deflection of the frame at the joint H due to a vertical load at joint I. Each member of the frame is of length ℓ and area A and Young's modulus E.

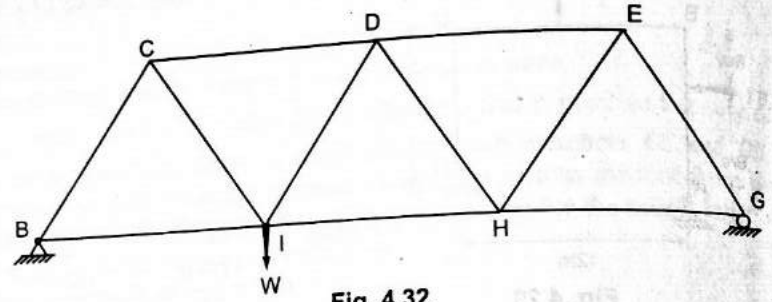


Fig. 4.32

[Ans. $\frac{5W\ell}{3AE}$]

5

INFLUENCE LINES FOR SIMPLE STRUCTURES

5.1 INTRODUCTION

In the case of static loads, bending moment and shear force diagrams are drawn by using the principle of statics. An example of bending moment diagram of a simply supported beam loaded with a central point load is shown in Fig. (5.1).

It is evident from the diagram that the ordinate of the vertex of the triangle (C) represents the maximum value of the function ($B.M.$). Now if the load moves from the center to any other point, the vertex of the triangle also moves changing the triangle's shape.

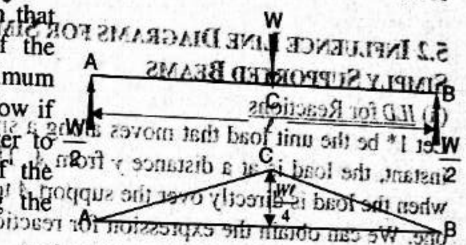


Fig. 5.1

If we consider the bending moment diagram for all the positions of the load, we will obtain a number of triangular diagrams as seen in Fig. (5.2). (e.g. ΔADB, ΔAEB…. etc) Joining the vertexes of all these triangular diagrams, we obtain a curve indicated by the dotted lines ($ABECFGB$). The vertical ordinate at any point to this curve from AB represents the maximum bending moment at the corresponding section when the load is on that particular section. The maximum ordinate under this curve represents the maximum of the entire maximum bending moments and is called absolute-bending moment. This is the most unfavorable situation for which a beam is designed under moving load condition.

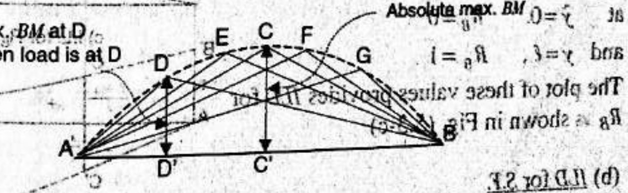


Fig. 5.2

But instead of this absolute bending moment, we may be interested to find the variation of bending moment/shear force over a particular section (say D) as a function of the position of the load. ΔADB in Fig. (5.2) represents $B.M.$

diagram for such variation. When such diagram is obtained for a moving load of unit magnitude, it is called influence line. Once influence lines diagram (*ILD*) is known, the value of a function for any load can be obtained by multiplying the ordinate of the diagram by the magnitude of load (in case of point load). Influence line can thus be defined as follows.

"An influence line is a curve, the ordinate to which at any point equals the value of some particular function due to a unit load acting at that point." The function may be bending moment, shear force or the reactions at supports. As maximum value of bending moment and shear force occur directly under the load, influence line diagram indicate how the maximum possible *B.M.* and *S.F.* at a section vary with the moving load of unit magnitude. Moreover, the ordinate of influence line is a dimensionless quantity.

5.2 Influence Line Diagrams for Simple Cases

Simply Supported Beams

(a) <u>*ILD* for Reactions</u>

Let 1* be the unit load that moves along a simply supported beam *AB*. At any instant, the load is at a distance y from *A*. This distance will vary from zero, when the load is directly over the support *A* to ℓ when it is over the right hand one. We can obtain the expression for reaction R_A by taking moment about *B*. Note that y is variable and x is constant here.

$\sum M_B = R_A \times \ell - 1 \times (\ell - y) = 0$

or, $R_A = \dfrac{\ell - y}{\ell}$ .... (5.1)

at $y = 0,\ R_A = 1$

at $y = \ell,\ R_A = 0$

Plotting these values, we get *ILD* for R_A as shown in Fig.(5.3-b).

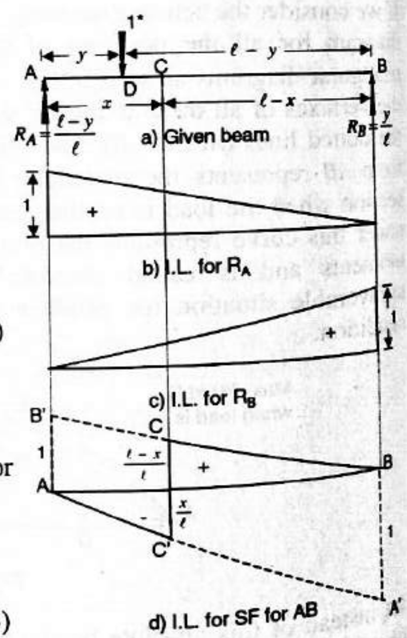
Again, $\sum F_y = 0,\quad R_A + R_B = 1$

or, $R_B = 1 - \dfrac{\ell - y}{\ell} = \dfrac{y}{\ell}$ ....(5.2)

at $y = 0,\quad R_B = 0$

and $y = \ell,\quad R_B = 1$

The plot of these values provides *ILD* for R_B as shown in Fig. (5.3-c)



(b) <u>*ILD* for *S.F.*</u>

for $y \le x$, $F_x = R_B = \dfrac{y}{\ell}$ ....(5.3)

at $y = 0,\quad F_x = 0$

and $y = x,\quad F_x = -\dfrac{x}{\ell}$

for $y \ge x$

$F_x = R_A = \dfrac{\ell - y}{\ell}$ ...... (5.4)

at $y = x,\quad F_x = \dfrac{\ell - x}{\ell}$

and $y = \ell,\quad F_x = 0$

The plot of these values gives *ILD* for *SF* as shown in Fig. (d). Now, in the figure, if line *BC* is extended to *B'* and *AC* to *A'*, we get, *AB'* = *BA'* = 1. These are obtained by substituting $y = \ell$ and $y = 0$ in Eq. (5.3) and (5.4) respectively.

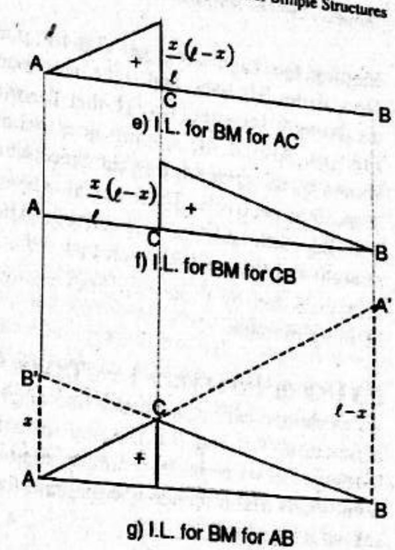


Fig. 5.3

Thus one can draw influence lines for shear force by plotting the ordinates +1 (upwards) and -1 (downwards) along the vertical passing through the left and right hand supports respectively and by joining each of the two points so obtained with the base point at the other support. After this, a vertical is traced through the section under consideration as in Fig. (d).

(c) <u>Influence line diagram for *B.M.*</u>

We have, $M_x = R_B(\ell - x)$ for $y \le x$

Which gives, $M_x = \dfrac{y}{\ell}(\ell - x)$ ....(5.5)

at $y = 0,\quad M_x = 0$

at $y = x,\quad M_x = \dfrac{x}{\ell}(\ell - x)$

Plotting these values, we get *ILD* for portion *AC* and it is shown in Fig. (e)

Similarly for $y > x$

$M_x = R_A x$, but $R_A = \dfrac{\ell - y}{\ell}$

$\therefore M_x = \dfrac{\ell - y}{\ell} x$ ....(5.6)

for $y = x$, $M_x = \dfrac{\ell - y}{\ell} x$

for $y = \ell$, $M_x = 0$

Plotting the values, we get *ILD* for portion *BC* and it is shown in Fig. (f). Now if the left hand and right hand portions of the influence line diagrams are brought together, we see that the ordinate at x coincide and thus obtain Fig. (g). Also if the sloping lines *BC* and *AC* are extended to *B'* and *A'* (as shown by the dotted lines), they meet at ordinates x and $(\ell - x)$ over supports *A* and *B* respectively. These can also be obtained by substituting $y = 0$ and $y = \ell$ in Eq. (5.6) and (5.5) respectively. Alternatively, we can also draw the two straight lines *AA* and *BB* such that $AB' = x$ and $BA' = (\ell - x)$ and the common area enclosed by these two lines represents the influence line diagram for bending moment.

5.3 Use of Influence Line Diagram

As explained earlier, influence line diagram is used to determine the value of a function (*S.F.* and *B.M.*) at any instant at any section due to any type of loading. Let us consider a simply supported beam loaded with a number of point loads. Its influence line diagram for *S.F.* and *B.M.* at a section *C* are also shown in fig. (5.4).

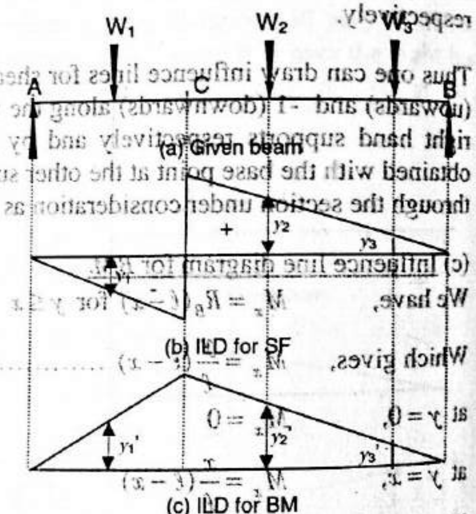


Fig. 5.4

Let y_1, y_2, y_3 and y'_1, y'_2, y'_3 are the ordinates of the influence line diagrams for *S.F.* and *B.M.* corresponding to the point of application of the loads W_1, W_2 and W_3.

Then shear force at *C* (F_C) and the bending moment (M_C) due to the loading are given by

$$F_C = -W_1 y_1 + W_2 y_2 + W_3 y_3$$

$$M_C = W_1 y'_1 + W_2 y'_2 + W_3 y'_3$$

Thus *S.F.* and moment values are the sum of multiplication of the magnitude of the loads and the ordinate of their corresponding *ILD* diagrams.

This is true in the case of point loads being applied. If the load is uniformly distributed, then the value of a function is determined by multiplying the intensity of loading (w) and the area of influence line diagram covered by the length of the loading in the beam.

Let a simply supported beam is acted on by a uniformly distributed load which is shorter in length than the span of the beam. At any instant, the moving load (*udl*) takes a position as shown in the Fig (5.5-a).

At this particular position of the load, we can find the value of *S.F.* and *B.M.* at the section *E*. For this, first we will have to draw influence lines diagrams for *S.F.* and *B.M.* for the section *E* shown in the Fig. (5.6-b) and (5.6-c).

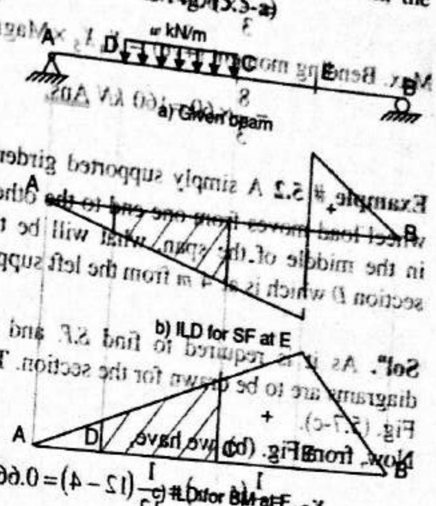


Fig. 5.5

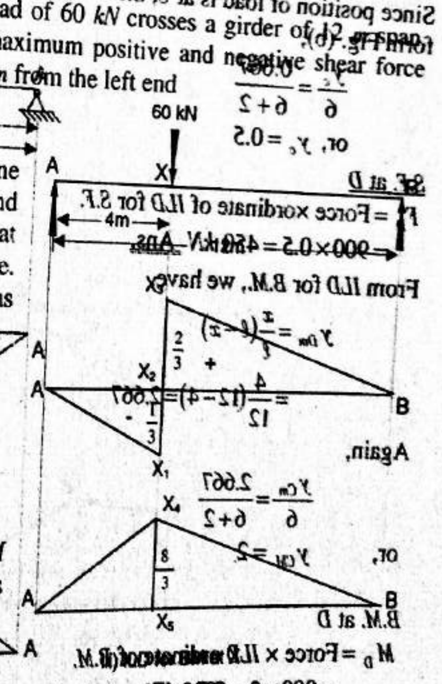
Now, *S.F.* at *E*,

$F_E = w \times$ shaded area from figure (a)

B.M. ate *E*, $M_E = w \times$ shaded area from figure (b)

Example # 5.1 A single point load of 60 *kN* crosses a girder of 12 *m* span. Using influence lines, find the maximum positive and negative shear force and bending moment at a point 4 *m* from the left end

Sol.



Let us first draw influence line diagrams for shear force and bending moment for the section at 4*m* from the left as shown above. The ordinates are computed as shown below.

Ordinate $X_1X_2 = \dfrac{x}{\ell} = \dfrac{4}{12} = \dfrac{1}{3}$ and

$X_2X_3 = \left(\dfrac{\ell - x}{\ell}\right) = \dfrac{12-4}{12} = \dfrac{8}{12} = \dfrac{2}{3}$

We know that maximum *SF* and *BM* at a section occur when the load is on the section itself.

Max. positive $SF = X_1X_2 \times$ Magnitude of the load

$$= \frac{1}{3} \times 60 = 20\,kN \text{ Ans.}$$

Max. negative $SF = X_2X_3 \times$ Magnitude of the load

$$= \frac{2}{3} \times 60 = 40\ kN\ \underline{\text{Ans.}}$$

Max. Bending moment (+ve) $= X_4X_5 \times$ Magnitude of the load

$$= \frac{8}{3} \times 60 = 160\ kN\ \underline{\text{Ans.}}$$

Example # 5.2 A simply supported girder has a span of $12\,m$. A $900\,kN$ wheel load moves from one end to the other. At the instant when the load is in the middle of the span, what will be the values of *S.F.* and *B.M.* at the section *D* which is at 4 *m* from the left support *A*.

Sol^n. As it is required to find *S.F.* and *B.M.* at section *D*, influence line diagrams are to be drawn for the section. They are shown in Fig. (5.7-b) and Fig. (5.7-c).

Now, from Fig. (b), we have,

$$y_{D1} = \frac{1}{\ell}(\ell - z) = \frac{1}{12}(12-4) = 0.667$$

Since position of load is at *C*, the ordinate of *ILD* at the section is required. So form Fig. (b),

$$\frac{y_c}{6} = \frac{0.667}{6+2}$$

or, $y_c = 0.5$

<u>*S.F.* at *D*</u>

$F_D =$ Force $\times$ordinate of *ILD* for *S.F.*

$= 900 \times 0.5 = 450\ kN$ <u>Ans.</u>

From *ILD* for *B.M.*, we have,

$$y_{Dm} = \frac{z}{\ell}(\ell - z)$$

$$= \frac{4}{12}(12-4) = 2.667$$

Again,

$$\frac{y_{Cm}}{6} = \frac{2.667}{6+2}$$

or, $y_{Cm} = 2$

<u>*B.M.* at *D*</u>

$M_D =$ Force $\times$ *ILD* ordinate of *B.M.*

$= 900 \times 2 = 1800\ kN - m$ <u>Ans.</u>

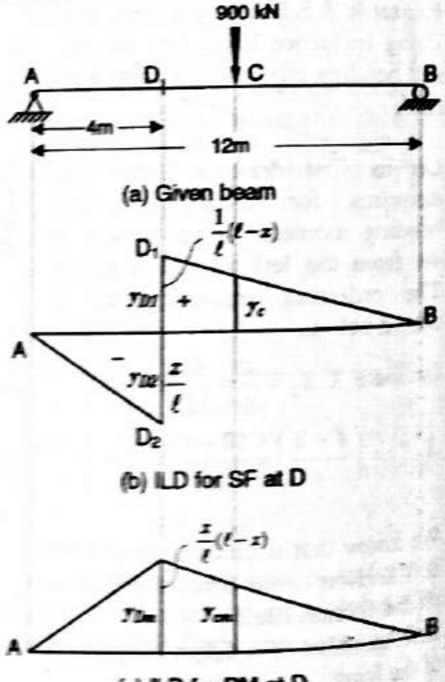


Fig. 5.7

Discussion

It is to be noted that the value of *S.F.* and *B.M.* in the solved example above are not the maximum values when the load moves from *A* to *B*. The maximum values will be obtained when the load reaches exactly on *D* i.e. at the place of maximum ordinates of *ILD*. The analysts are often interested to determine the maximum values of functions (*S.F.* and *B.M.*) for a particular section, when the load moves the entire span. The first step in such case is then to find the position of the load to yield maximum *S.F.* and *B.M.* For a single point load, as in the above example, maximum value of a function occurs when the load reaches the section itself. However, problem arises to fix the load position when there are numbers of loads moving on the span. Problems of such nature are dealt in the following sections.

There is yet another type of problem when the section for maximum values of functions is not specified. In such case, one has to find the section or the place where maximum *S.F.* and *B.M.* occurs before finding the load position. There are dealt systematically in the following topics

Example # 5.3 Two point loads of 80 *kN* and 160 *kN* spaced 2 *m* apart, cross a girder of span 10 *m* with the 80 *kN* load leading from left to right. Draw influence lines for *S.F.* and *B.M.* and find the value of maximum *S.F.* and *B.M.* at a section 4 *m* from the left end support.

Sol^n.

Leading load, $W_2 = 80\ kN$

Trailing load, $W_1 = 160\ kN$

Spacing of load $d = 2\ m$

Span $\ell = 10\ m$

(a) Positive *S.F.*

First, draw the influence lines for the positive *SF.*

The ordinate XX_2 is

$$\frac{\ell - z}{\ell} = \frac{6}{10} = 0.6$$

From the similar triangles EXB and CDB, $CD = 0.4$

The maximum positive *S.F.* at *X* takes place, when the trailing load is at *X*

$\therefore F_{max} = W_1 \times CD + W_2 \times XX_2$

$= 80 \times 0.4 + 160 \times 0.6 = 128\ kN$ <u>Ans.</u>

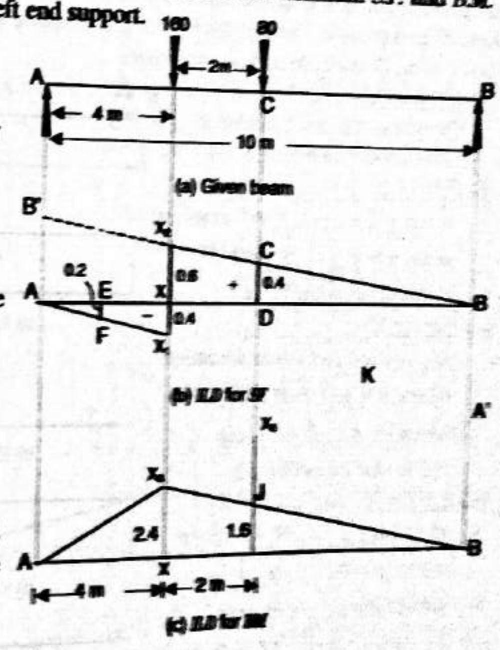


Fig. 5.8

(b) Negative shear force

From Fig. (b), XX_2 equal to $x/l = 4/10$

From the geometry of the figure, we find that $EF = 0.2$

We know that maximum negative S.F. at X takes place, when the trailing load is on the section.

$\therefore F_{max} = W_1 \times XX_1 + W_2 \times EF = 80 \times 0.4 + 160 \times 0.2 = 64$ kN Ans.

(c) Bending moment

Now, draw the influence lines for the B.M. on the base AB with ordinate XX equal to $\frac{x(\ell - x)}{\ell} = \frac{4(10-4)}{10} = 2.4$

From the geometry of the figure, we find that $JK = 1.6$

We know that maximum B.M. at X, takes place when the trailing load is at X and the leading load is in XB.

$\therefore M_{max} = 160 \times X_3X + 80 \times JK = 160 \times 2.4 + 80 \times 1.6$

$= 512$ kN-m Ans.

5.4 Influence Line Diagrams for Cantilever Beams

Let us consider a cantilever beam acted on by a unit-moving load over its span of length ℓ. Let R_A, F_A and M_A be the reaction, shear force and bending moment of the beam whose influence lines are to be drawn.

Let, at an instance, the unit load 1 is at a distance y from B. It is to be noted that y here, is variable and x is constant.

(a) ILD for F_A

The shear force = 1 constant

ILD is shown in Fig. (b)

(b) ILD for F_c

when $y < x$, $F_c = 1$ constant

when $y > x$, $F_c = 0$

ILD is shown in Fig. (c)

(c) ILD for M_A

$M_A = (\ell - y) \rightarrow$ linear variation

when $y = 1$, $M_A = \ell$

when $y = \ell$, $M_A = 0$

ILD is shown in Fig. (d)

(d) ILD for M_c

or $y \le x$, $M_c = -1(x - y)$

when $y = 0$, $M_c = -x$

when $y = x$, $M_c = 0$

for $y > x$, $M_c = 0$ constant

ILD is shown in Fig. (e)

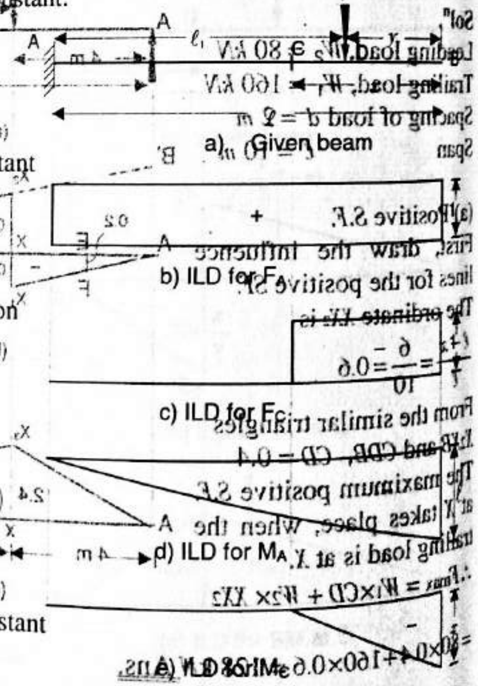


Fig. 5.9

5.5 Influence Line Diagrams for Overhanging Beam

(a) ILD for R_B

Referring Fig. (b)

when load is in BC, $\Sigma M_A = 0$

$R_B \times \ell = 1 \times (\ell + a - x)$

or, $R_B = 1 + \frac{a - y}{\ell}$

at, $y = 0$, $R_B = \frac{\ell + a}{\ell}$

at, $y = a$, $R_B = 1$

Referring Fig. (c)

when load is at AB,

$R_B = \frac{\ell - y}{\ell}$

at $y = 0$, $R_B = 1$

at $y = \ell$, $R_B = 0$

ILD is shown is Fig. (d)

(b) ILD for R_A

Referring Fig. (b)

when load is at BC,

$R_A = R_B - 1 = \frac{a - y}{\ell}$

(R_A acts downward, –ve)

at $y = 0$, $R_A = \frac{a}{\ell}$

at $y = a$, $R_A = 0$

Referring Fig. (c)

when load is in AB,

$R_A = \frac{y}{\ell}$

at $x = 0$, $R_A = 0$

at $y = \ell$, $R_A = 1$

(R_A is upward +ve)

ILD is shown in Fig. (e)

(c) ILD for shear force at D (F_D)

Referring Fig. (b),

when load is in BD, $F_D = 1$

when load is in BD, $F_D = 0$

Referring Fig. (c), when load is at AB,

$F_D = 0$, ILD is shown in Fig. (f)

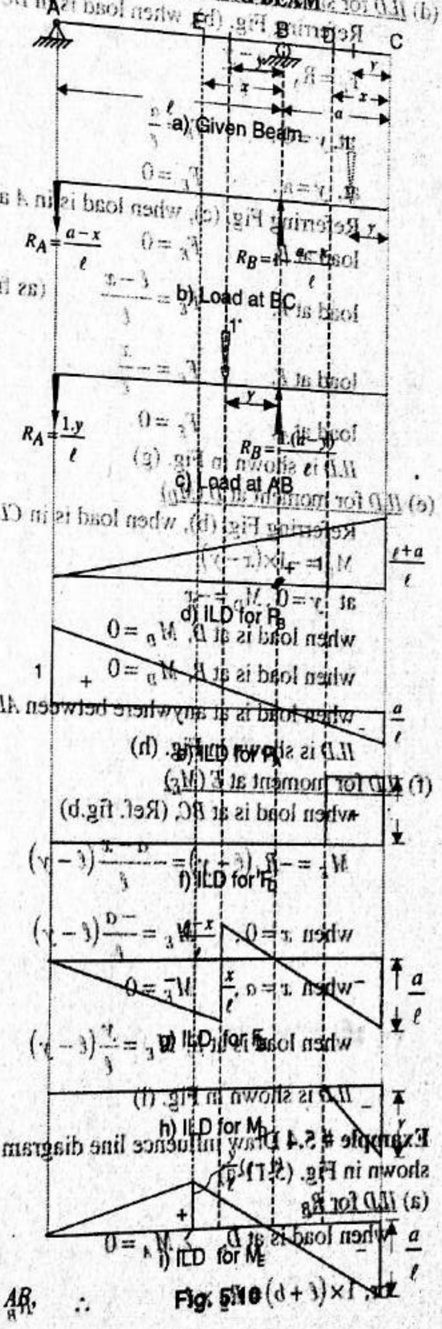


Fig. 5.10

(d) <u>*ILD* for shear force at E (F_E)</u>

Referring Fig. (b), when load is in *BC*

$F_E = R_A = \dfrac{a-y}{\ell}$

at, $y=0$, $F_E = \dfrac{a}{\ell}$

at, $y=a$, $F_E = 0$

Referring Fig. (c), when load is in *A* and *B*

load at *B*, $F_E = 0$

load at *E*, $F_E = \dfrac{\ell - x}{\ell}$ (as from previous examples)

load at *E*, $F_E = -\dfrac{x}{\ell}$

load at *A*, $F_E = 0$

ILD is shown in Fig. (g)

(e) <u>*ILD* for moment at *D* (M_D)</u>

Referring Fig. (b), when load is in *CD*,

$M_D = -1\times(x-y)$

at $y=0$, $M_D = -x$

when load is at *D*, $M_D = 0$

when load is at *B*, $M_D = 0$

when load is at anywhere between *AB*, $M_D = 0$

ILD is shown in Fig. (h)

(f) <u>*ILD* for moment at *E* (M_E)</u>

when load is at *BC*, (Ref. fig.b)

$M_E = -R_A(\ell - y) = -\dfrac{a-x}{\ell}(\ell - y)$

when $x=0$, $M_E = \dfrac{-a}{\ell}(\ell - y)$

when $x=a$, $M_E = 0$

when load is at *E*, $M_E = \dfrac{y}{\ell}(\ell - y)$ (as from previous examples)

ILD is shown in Fig. (i)

Example # 5.4 Draw influence line diagram for R_A, R_B, F_E and M_E of the beam shown in Fig. (5.11-a)

(a) <u>*ILD* for R_B</u>

when load is at *D*, $\sum M_A = 0$

or, $1\times(\ell + b) = R_B \times \ell$ $\therefore$ $R_B = \dfrac{\ell + b}{\ell}$

when load is at *B*, $R_B = 1$

when load is at *A*, $R_B = 0$

when load is at *C*, $\sum M_A = 0$

or, $R_B \times \ell = 1\times a$

or, $R_B = \dfrac{a}{\ell}$ (downward)

ILD is shown in Fig. (b)

(b) <u>*ILD* for R_A</u>

when load is at *D*, $\sum M_B = 0$

or, $1\times b = R_A \times \ell$

or, $R_A = \dfrac{b}{\ell}$ (downward)

when load is at *B*, $R_A = 0$

when load is at *A*, $R_A = 1$

when load is at C, $\sum M_B = 0$

$1\times(\ell + a) = R_A \times \ell$

or, $R_A = \dfrac{\ell + a}{\ell}$

ILD is shown in Fig. (c)

(c) <u>*ILD* for F_E</u>

when load is at *D*,

$F_E = R_A = \dfrac{-b}{\ell}$, when

load is at *B*, $F_E = 0$

when load is at *E*, (just left of *E*), $F_E = -R_B = \dfrac{-y}{\ell}$

when load is at *E*, (just right of *E*), $F_E = R_A = \dfrac{\ell - y}{\ell}$

when load is at *A*, $F_E = 0$ and at *C*, $F_E = R_B = \dfrac{a}{\ell}$ *ILD* shown in Fig. (d)

(d) <u>*ILD* for M_E</u>

when load is at *D*, $M_E = R_A(\ell - y) = \dfrac{b}{\ell}(\ell - y)$

when load is at *B*, $M_E = 0$

when load is at *E*, $M_E = \dfrac{y}{\ell}(\ell - y)$ (as in previous examples)

when load is at *C*, $M_E = R_B \times y = -\dfrac{a}{\ell}.y$

ILD is shown in Fig.(e)

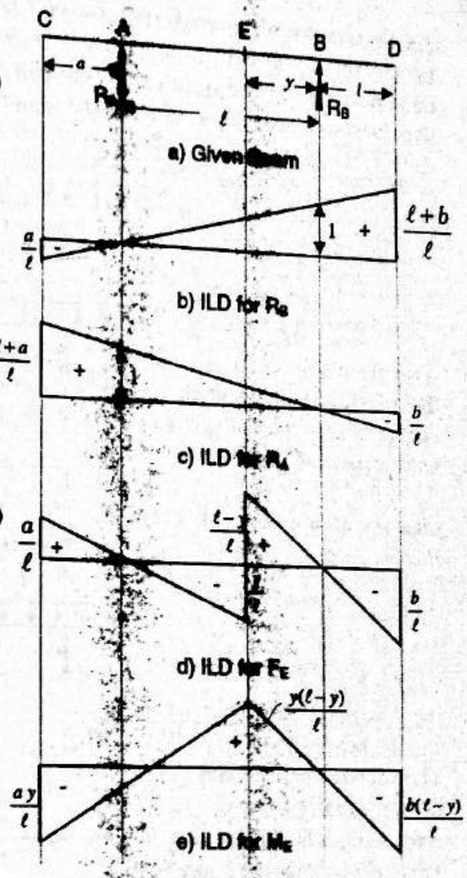


Fig. 5.11

5.6 UNIFORMLY DISTRIBUTED LOAD LONGER THAN THE SPAN

(a) Maximum S.F. and B.M. at a section: From the ILD for S.F. it is clear that max. positive S.F. will occur when the span AC is loaded and CB is empty and max. negative S.F. will occur when the span CB is loaded and AC is empty.

$$\therefore +F_x = w.\frac{1}{2}.x.\frac{x}{\ell} = \frac{wx^2}{2\ell}$$

$$-F_x = w.\frac{1}{2}.(\ell - x)\left(\frac{\ell - x}{\ell}\right)$$

$$= -\frac{w(\ell - x)^2}{2\ell}$$

Also from Fig. (f) it is clear that the maximum moment at C will occur, when the udl covers the entire span.

max +ve $M_x = w \times$ area of +ve Δ of ILD.

$$= w.\frac{1}{2}.\ell.\frac{x}{\ell}(\ell - x)$$

$$= \frac{wx(\ell - x)}{2}$$

(b) Abs max. values anywhere in the beam

The above expressions of ILD shows that positive S.F. is maximum, when $x = \ell$ i.e. when the load occupies the entire span.

ordinate of positive ILD at $x = \ell = 1$

$$\therefore +F_{abs} = w \times \frac{1}{2} \times \ell \times 1 = \frac{w\ell}{2}$$

Similarly, $-F_{abs} = w \times \frac{1}{2} \times \ell \times 1 = \frac{w\ell}{2}$

Max. +ve moment occurs at centre

where $x = \frac{\ell}{2}$

ordinate of $ILD = \ell/4$

$$\therefore M_{abs} = w.\frac{1}{2} \times \ell \times \frac{\ell}{4} = \frac{w\ell^2}{8}$$

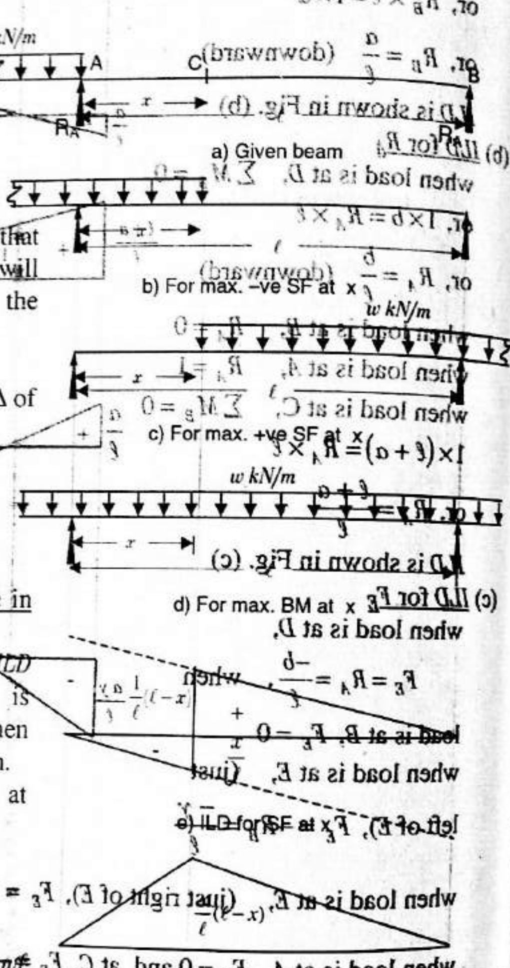


Fig. 5.12

5.7 UNIFORMLY DISTRIBUTED LOAD SHORTER THAN SPAN

(a) Max. S.F. and B.M. at a section

As seen from ILD, maximum negative shear force occur at the section x when head of the load touches at it and the maximum positive shear force occurs when the tail touches the section. The maximum values are obtained by multiplying intensity of the load by the respective shaded areas. Maximum bending moment at x will occur when the load is partly on AC and CB as shown in Fig. (5.13-e). Let a be the length of the load and z be the distance of the tail of the load to the section as shown in the Fig. (5.13-e).

Then, $M_c = w.\frac{(y_1 + y_c)}{2}.z + w.\frac{(y_c + y_2)}{2}(a - z)$

(a is the variable here)

For maximum value of M_c

$$\frac{dM_c}{dz} = w\frac{(y_1 + y_c)}{2} - w\frac{(y_c + y_2)}{2} = 0$$

Which gives, $y_1 = y_2$ ....(5.7)

Now, we have,

$$\frac{y_1}{y_c} = \frac{x - z}{x}$$

or, $y_1 = \frac{x - z}{x}.y_c$

and, $\frac{y_2}{y_c} = \frac{(\ell - x) - (a - z)}{(\ell - x)}$

or, $y_2 = \frac{(\ell - x) - (a - z)}{(\ell - x)}.y_c$

Substituting values of y_1 and y_2 in Eq. (5.7) we get,

$$\frac{x - z}{x}.y_c = \frac{(\ell - x) - (a - z)}{(\ell - x)}.y_c$$

or, $\frac{x - z}{x} = \frac{\ell - x - a + z}{\ell - x}$

or, $(x - z)(\ell - x) = \ell x - x^2 - ax + zx$

or, $\ell z = ax$

Which gives $\frac{x}{\ell} = \frac{z}{a}$ ....(5.8)

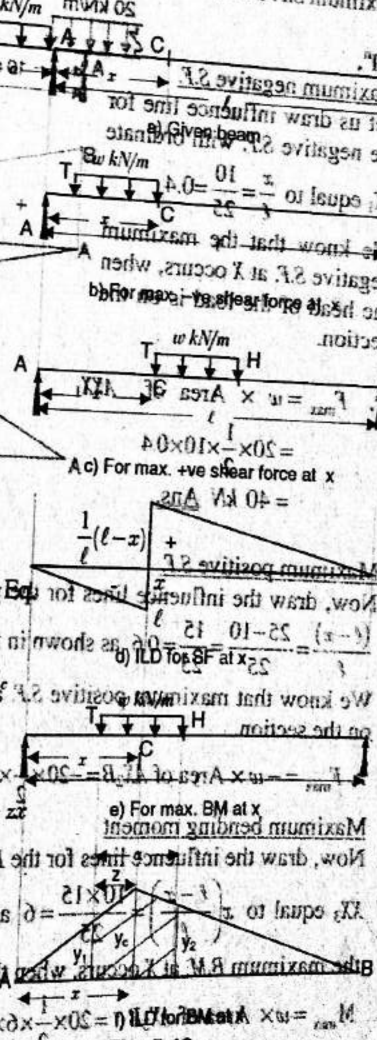


Fig. 5.13

It shows that maximum bending moment occur at a section when the section divides the load in the same ratio as it divides the span. By using Eq. (5.8), we can easily find the value of x, which is needed for finding the max. $B.M.$ at x.

(i) Position of absolute maximum moment
Absolute maximum moment will occur, when $x=\frac{\ell}{2}$ This is proved in section 5.8- Eq. (5.9).

Example # 5.5 A uniformly distributed load of 20 kN/m, longer than span, rolls over a beam of 25 m span. Using influence lines determine the maximum $S.F.$ and $B.M.$ at a section 10 m from the left end support.

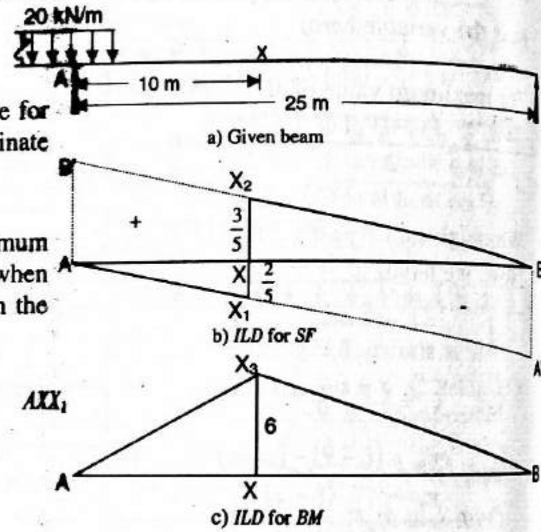


Fig. 5.14

Sol^n.

Maximum negative $S.F.$
Let us draw influence line for the negative $S.F.$ with ordinate

XX_1 equal to $\frac{x}{\ell}=\frac{10}{25}=0.4$

We know that the maximum negative $S.F.$ at X occurs, when the head of the load is on the section.

$$\therefore\ F_{max} = w \times \text{Area of } AXX_1$$
$$=20\times\frac{1}{2}\times10\times0.4$$
$$=40\ kN\ \underline{\text{Ans.}}$$

Maximum positive $S.F.$
Now, draw the influence lines for the positive $S.F.$ with ordinate XX_2 equal to $\frac{(\ell-x)}{\ell}=\frac{25-10}{25}=\frac{15}{25}=0.6$ as shown in figure.

We know that maximum positive $S.F.$ at X occurs, when the tail of the load is on the section.

$$\therefore F_{max}=-w\times \text{Area of } AX_2B=-20\times\frac{1}{2}\times0.6\times15=-90\ kN$$

Maximum bending moment
Now, draw the influence lines for the $B.M.$ on the base AB with central ordinate XX_3 equal to $x\left(\frac{\ell-x}{\ell}\right)=\frac{10\times15}{25}=6$ as shown in Fig.(5.14-c). We know that the maximum $B.M.$ at X occurs, when the entire span is fully loaded.

$$M_{max}=w\times \text{Area of } AX_3B\ =20\times\frac{1}{2}\times6\times25=1500\ kN-m\ \underline{\text{Ans}}$$

Example # 5.6 A uniformly distributed load of 50 kN/m of 6 m length crosses a girder of span 40 m from left to right. With the help of influence lines, determine the values of $S.F.$ and $B.M.$ at a point 12 m from the left support when the head of the load is 16 m from the left support.

Sol^n.

Shear Force
Let us draw the influence lines for the positive and negative $S.F.$ with ordinate XX_1 equal to $\frac{x}{\ell}=\frac{12}{40}=0.3$ and XX_2 equal to $\frac{(\ell-x)}{\ell}=\frac{40-12}{40}=\frac{28}{40}=0.7$

From the geometry of the figure,

$$CD=0.3\times\frac{10}{2}=\frac{3}{12}=\frac{1}{4}=0.25$$

and $EF=0.7\times\frac{24}{28}=\frac{6}{10}=0.6$

We know that the $S.F.$ at X_1

$$F_x=w\left[Area\ CDX_1X - Area\ EFX_2X\right]$$

$$=50\begin{bmatrix}2\times\frac{1}{2}(0.3+0.25)\\-4\times\frac{1}{2}(0.7+0.6)\end{bmatrix}kN$$

$$=50\left[0.55-2.6\right]kN=102.5\ kN$$

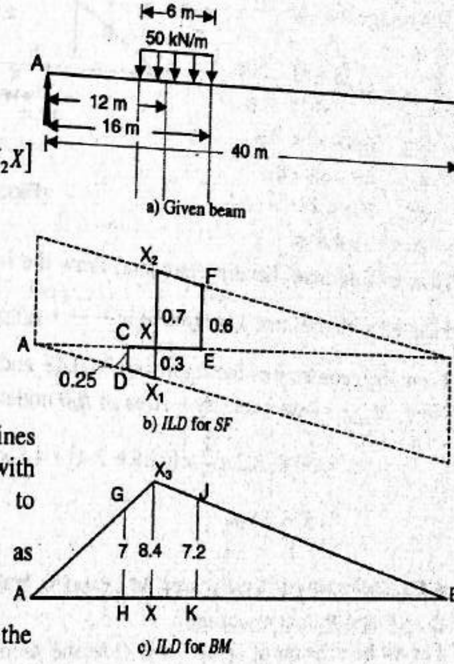


Fig. 5.15

Bending Moment
Now, draw the influence lines for $B.M.$ on the base AB with ordinate XX_3 equal to $x\frac{(\ell-x)}{\ell}=12\frac{(40-12)}{40}=8.4$ as shown in figure (c)

From the geometry of the figure, we find that

$$CH=8.4\times\frac{10}{12}=7$$

$$JK=8.4\times\frac{24}{28}=7.2$$

We know that the $B.M.$ at X, $M_x=w(Area\ GHXX_3+Area\ JKXX_3)$

$$=50\left[2\times\frac{1}{2}\times(7+8.4)+4\times\frac{1}{2}(7.2+8.4)\right]kN\text{-}m$$

$$=2330\ kN\text{-}m.\ \underline{\text{Ans.}}$$

Example # 5.7 A live load of 50 *kN/m*, 8 *m* long moves on a simply supported girder of span 10 *m*. Find the maximum bending moment which can occur at a section 4 *m* from left end.

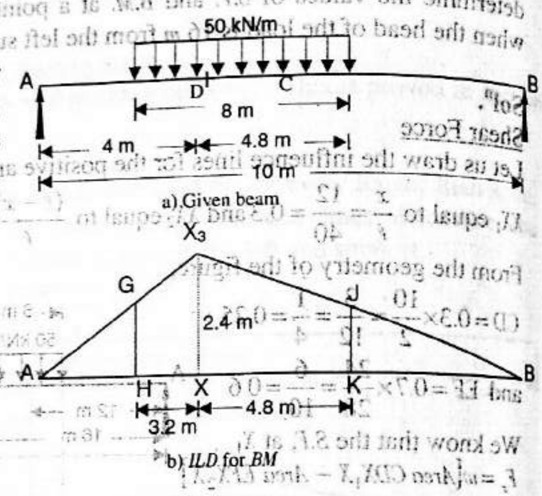


Fig. 5.16

Sol^n. Let AB be the girder and C be the given section. Let head of the load is at a distance z from the section D and it moves towards the right.

Average load on AD

= Average load on BD

$$50\frac{(8-z)}{4}=\frac{50z}{6}$$

or, $6(8-z)=4z$

or, $48-6z=4z$

or, $10z=48$

$\therefore$ $z=4.8\ m$

Now to find max. bending moment, draw the influence lines for $B.M.$ on the base AB with ordinate XX_3 equal to $x\dfrac{(\ell-x)}{\ell}=4\dfrac{(10-4)}{10}=2.4\ m$

From the geometry of the figure, $GH=0.48$ and $JK=0.48$

Now, M_{max} = load intensity × Area of ILD under the load

$$=50\times\left[3.2\times\frac{1}{2}\times(0.48+2.4)+4.8\times\frac{1}{2}\times(0.48+2.4)\right]$$

$$=576\ kN\text{-}m$$

5.8 Concept of Absolute Maximum Values

(a) S.F. and B.M. at a section

Let AB be a beam of span ℓ and C be the section anywhere in the span. It is clear from the previous section that to find the maximum $S.F.$ and $B.M.$, we first have to draw influence line diagrams for the particular section. The diagrams are shown in the Fig. (5.17-b and c). If W is the magnitude of the load, then as the maximum ordinate of the influence line is at the section itself.

Max. +ve $S.F.$ at x,

$$F_x=W.\frac{1}{\ell}(\ell-x)$$

Max. –ve $S.F.$ at x,

$$F_x=W.\frac{x}{\ell}(\ell-x)$$

It is noted that these values are for the particular moment of time, when the rolling load is directly over the section under consideration.

(b) Maximum S.F. and B.M. anywhere in the beam (Absolute max. S.F. & B.M.)

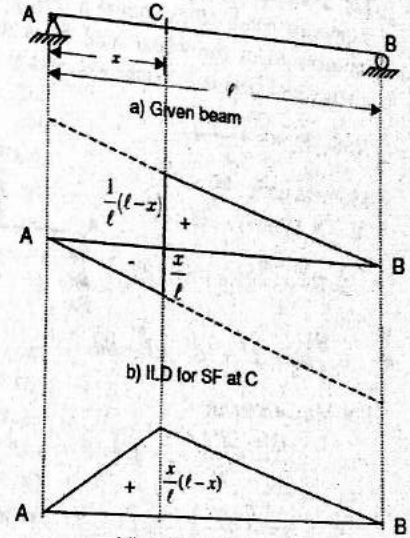


Fig. 5.17

A little consideration shows that the maximum ordinate for +ve $S.F.$ is obtained when $x=0$ in the expression $(\ell-x)/\ell$. Similarly, max. –ve $S.F.$ is obtained when $x=\ell$ in the expression x/ℓ. Their values are ±1.

$\therefore$Max. +ve $S.F.$ $F_{abs}=W\times(1)=W$

Max. –ve $S.F.$ $F_{abs}=W(-1)=-W$

These are the values for $S.F.$ at A and B when the loads are directly over these locations.

Similarly, maximum ordinate of ILD for BM is obtained at the place where

$$\frac{d\left[\frac{x}{\ell}(\ell-x)\right]}{dx}=0$$

i.e. $1-\dfrac{2x}{\ell}=0$

or, $x=\dfrac{\ell}{2}$ (i.e. at the centre) .... (5.9)

ILD ordinate at $x=\dfrac{\ell}{2}$ is $\dfrac{\ell/2}{\ell}\left(\ell-\dfrac{\ell}{2}\right)=\dfrac{\ell}{4}$

$\therefore$ Max. +ve $B.M$, $M_{abs}=w\times\ell/4$ (where the load is at the centre of the beam)

These values are the maximum possible values in the beam and are also called absolute maximum $B.M.$ and $S.F.$

It is convenient to use this principle to find absolute maximum values for beams subjected to several point loads, which will be illustrated in the later sections.

5.9 Several Point Loads

(a) Maximum Moment under a Load

Let a set of several concentrated loads $W_1, W_2, W_3, \ldots$etc. move on a simply supported beam AB as shown in Fig. (5.18). Let the condition for maximum moment under the wheel load W_2 is required. Let R be the resultant of all loads which acts at x distance from A .

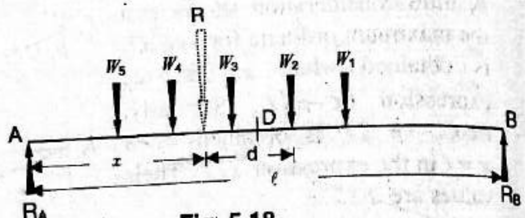


Fig. 5.18

Now, $R_A = \dfrac{R(\ell - x)}{\ell}$

Moment under W_2

$$M = R_A(x+d) - Rd$$

$$= R\frac{(\ell - x)}{\ell}(x+d) - Rd$$

$$= \frac{R}{\ell}\left[\ell x - x^2 + \ell d - xd\right] - Rd$$

For Max. moment,

$$\frac{dMx}{dx} = \frac{R}{\ell}[\ell - d - 2x] = 0$$

$$0 = \frac{R}{\ell}[\ell - d - 2x]$$

or, $\ell - d - 2x = 0$

or, $2x = \ell - d$

or, $x = \dfrac{(\ell - d)}{2}$ .... (5.10)

Distance of W_2 from $A = x + d$

$$= \frac{(\ell - d)}{2} + d$$

$$= \frac{\ell}{2} - \frac{d}{2} + d$$

$$= \frac{\ell}{2} + \frac{d}{2}$$ .... (5.11)

From Eq. (5.10) and Eq. (5.11), we can conclude that for maximum moment under any particular load the position of the load and the resultant should be equidistant from the mid. span.

(b) Maximum shear force and bending moment at a section

Let us consider a beam on which a number of loads are crossing as shown in Fig. (a). For the maximum value of positive and negative $S.F.$, the loads are made to coincide alternatively at the section and their values are calculated. A number of trials may be necessary to find the maximum value depending upon the number of loads being applied. It is noted that for maximum negative shear force, most of the loads are to be left of the section. Similarly

maximum positive $S.F.$ occurs at the section when most of the loads are at the right of the section.

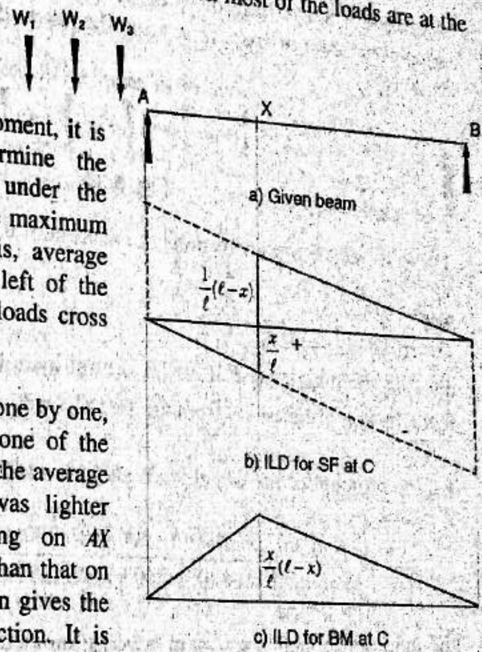


Fig. 5.19

In the case of bending moment, it is first necessary to determine the particular load position under the section x that will provide maximum bending moment. For this, average loading to the right and left of the section is studied as the loads cross the section X.

As they cross the section one by one, a condition arises when one of the loads crosses the section; the average loading on XB (which was lighter than the average loading on AX before) becomes heavier than that on AX. This particular position gives the maximum BM on the section. It is illustrated in the following examples.

(c) Absolute maximum shear force and bending moment

From the figure, as maximum ordinate are at the supports, the maximum $S.F.$ occurs when one of the load is at the support (W_1, W_2 or W_3 in the above Fig.(c) Few trials may thus be necessary to determine the maximum value. Obviously, maximum positive $S.F.$ will occur at A and the maximum negative $S.F.$ will occur at B.

In the case of absolute maximum bending moment, preposition given below is used.

"When a series of point loads crosses a girder, simply supported at its ends, the maximum $B.M.$ under any given load occurs when the centre of the span is mid way between the $C.G.$ of the load system and the load under which the maximum bending moment is required to be found out."

With the help of this preposition, the maximum moments under the possible loads placed at centre are evaluated and the maximum of these will then be the absolute maximum bending moment.

Example # 5.8 A train of wheel loads as shown in figure crosses a simply supported beam of span 25 *m.* from the left to the right with the 200 *kN* load leading.

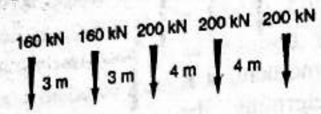


Fig. 5.20

Using influence lines determine the maximum *B.M.* under central load.

Sol^n.

Given Span, $\ell = 25\,m$

To find the maximum *B.M.* under central load, let the resultant of the wheel loads be at a distance $\bar{x}$ from the 160 *kN* load.

Taking moment of the wheel loads about the 160 *kN* load,

We have,

$$\therefore \bar{x} = \frac{(160\times3+200\times6+200\times10+200\times14)}{(160+200+200+200+160)}$$

$$= 7.04$$

For the condition of maximum bending moment under the central load, the wheel load system should be so placed that the resultant load and the central load should be equidistant from the middle point of the beam.

Let us place the load accordingly and determine the required distance x as shown in the Fig. (5.21).

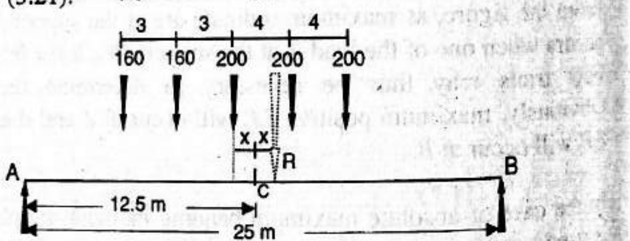


Fig. 5.21

The distance

$$x = \frac{(7.04-6)}{2} = 0.52\,m$$

Hence, bending moment under central load will be maximum when it is placed at 0.52 *m* left of the centre of the beam.

Now, taking moment about *A*, we have

$$R_B \times 25 = 920\times(12.5+0.52) \Rightarrow R_B = 479.14\,kN$$

$$\therefore \quad R_A = 920 - 479.14 = 440.86\ kN$$

$\therefore$ Max. bending moment under the central load

$$= 40.86\times11.98 - 160\times3\times160\times6 - 200\times8$$

$$= 3341.5\ \text{kN}-\text{m}\ \underline{\text{Ans.}}$$

Example # 5.9 A train of 5 wheel loads as shown in Fig. (5.22) crosses a simply supported beam of span 22.5 *m.* Calculate the maximum positive and negative shear force values at the centre of span and the absolute maximum *B.M.* anywhere in the span.

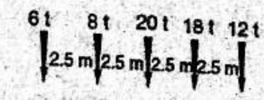


Fig. 5.22

Sol^n.

(a) <u>Maximum positive shear force</u>

Now, draw the influence lines for the positive shear force on the base *AB*, with ordinate XX_2 equal to $\frac{\ell - x}{\ell} = \frac{(22.5-11.25)}{22.5} = \frac{1}{2}$

Now, from geometry of the figure, we get,

$$f_8 f_8' = \frac{1}{18},\ f_7 f_7' = \frac{3}{18},\ f_6 f_6' = \frac{5}{18} \text{ and } f_5 f_5' = \frac{7}{18}$$

So, Maximum positive Shear force at *C* occurs, when the load is on the section itself.

$$F_{max} = \left[6\times XX_2 + 8\times f_5 f_5' + 20\times f_6 f_6' + 18\times f_7 f_7' + 12\times f_8 f_8'\right]$$

$$= \left[6\times\frac{1}{2} + 8\times\frac{7}{18} + 20\times\frac{5}{18} + 18\times\frac{3}{18} + 12\times\frac{1}{18}\right] = 15.33\,t\ \underline{\text{Ans.}}$$

(b) <u>Max. negative shear force</u>

First of all, draw the influence lines for the negative *S.F.* on the base *AB*, with the ordinate XX_1 equal to $\frac{x}{\ell} = \frac{11.25}{22.5} = \frac{1}{2}$ as shown in figure.

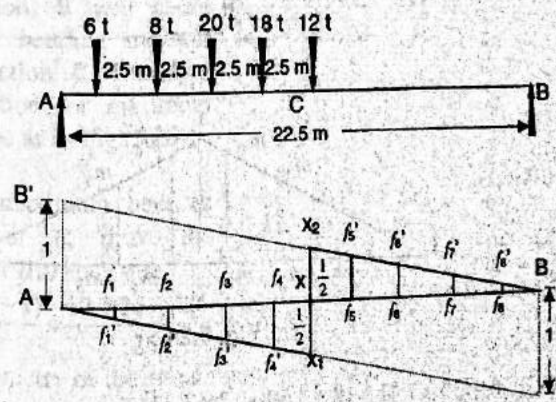


Fig. 5.23

From the geometry of the figure AXX_1, we find that

$f_1f_1' = \frac{1}{18}$, $f_2f_2' = \frac{3}{18}$, $f_3f_3' = \frac{5}{18}$, and $f_4f_4' = \frac{7}{18}$

We know that the maximum negative shear force at C occurs, when the loading is at C.

$$F_{max} = 12\times XX_1 + 18\times f_4f_4' + 20\times f_3f_3' + 8f_2f_2' + 6\times f_1f_1'$$

$$= 12\times\frac{1}{2} + 18\times\frac{7}{18} + 20\times\frac{5}{18} + 8\times\frac{3}{18} + 6\times\frac{1}{18} = 20.22\,t$$

(c) <u>Absolute maximum bending moment</u>

For this, let us first find out the $C.G.$ of the load system. $C.G.$ from $6t$ load is

$$\bar{x} = \frac{8\times2.5 + 20\times5 + 18\times7.5 + 12\times10}{8+20+18+12+6}$$

$$= 5.86\,m$$

By observation, we find that max. $B.M.$ occurs under $20t$ load. For this, the load should occupy a position on the beam such that the centre of the span is at mid way between CG of the load and the $20t$ load.

Therefore $20t$ load should be at a distance $11.25 - \frac{1}{2}(5.86 - 5.00) = 10.82\,m$ from A as shown in figure below.

Now draw the influence lines for the $B.M.$ on the base AB, with ordinate XX_3 equal to $\frac{x(L-x)}{\ell} = \frac{11.25(22.5-11.25)}{22.5} = 5.625$

From ΔAX_3B, we find that

$m_1m_1' = 2.91$, $m_2m_2' = 4.16$,

$m_3m_3' = 5.41$, $m_4m_4' = 4.59$

and $m_5m_5' = 3.34$

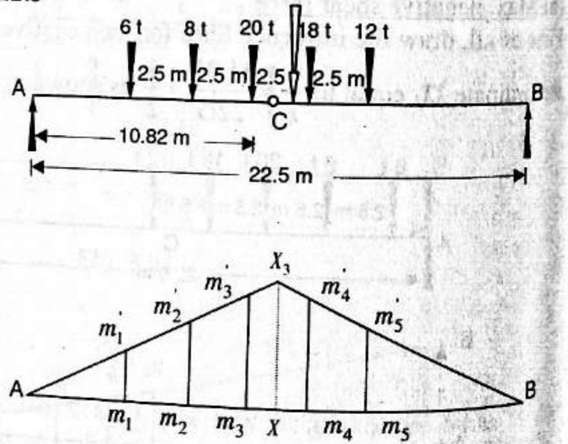


Fig. 5.24

$\therefore M_{abs}.\max$

$$= 6\times m_1m_1' + 8\times m_2m_2' + 20\times m_3m_3' + 18\times m_4m_4' + 12\times m_5m_5'$$

$$= 6\times2.91 + 8\times4.16 + 20\times5.41 + 18\times4.59 + 12\times3.34$$

$$= 281.61\,t-m$$ <u>Ans.</u>

Example # 5.10 A system of 5 loads of 80 kN, 160 kN, 160 kN, 60 kN and 40 kN crosses a beam of 25 m span with the 80 kN load leading. The distance between the loads is 2.4, 3.0, 2.4 and 1.8 m respectively. Find the maximum $B.M.$ at the centre of the span. Also, find the absolute maximum bending moment on the beam.

Sol^n. (a) <u>Maximum bending moment at the centre</u>

The maximum $B.M.$ at the centre of the span will occur when one of the load is at the centre itself. By guess we can say that when central load (160 kN) is at the centre, it will cause maximum $B.M.$ however, we can confirm this by studying average loading on the sections AC and CB. To do so, we allow the loads to cross the half section one by one and study the average loading on both sides of the section.

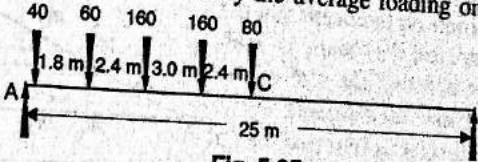


Fig. 5.25

Load Crossing the Section	Average Loading in AC	Average Loading in BC	Remarks
80 kN	$\frac{420}{12.5}$	$\frac{80}{12.5}$	Loading on AC is heavier than BC
160 kN	$\frac{260}{12.5}$	$\frac{240}{12.5}$	Loading on AC is heavier than BC
160 kN	$\frac{100}{12.5}$	$\frac{400}{12.5}$	Loading on AC is lighter than BC

It is thus obvious, that when the 160 kN load will be at the half section, It will cause maximum bending moment at the section C. Therefore load position for maximum BM will be as in Fig.(5.26).

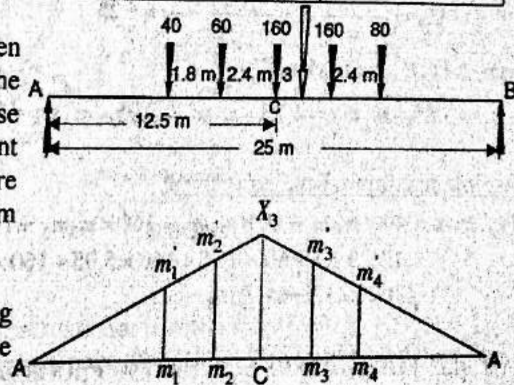


Fig. 5.26

To find maximum bending moment at C, draw the ordinate of ILD such that,

$$CX_3 = \frac{x(\ell-x)}{\ell} = \frac{12.5(25-12.5)}{25} = 6.25\,m$$

From Geometry of the given figure, we get,

$m_1m_1' = 4.15$, $m_2m_2' = 5.05$, $m_3m_3' = 4.75$, and $m_4m_4' = 3.55$

To find out Maximum moment

$$M_{max} = 400 \times m_1 m_1' + 60 \times m_2 m_2' + 160 \times X_3 C + 160 \times m_3 m_3' + 80 \times m_4 m_4'$$
$$= 400 \times 4.15 + 60 \times 5.05 + 160 \times 6.25 + 160 \times 4.75 + 80 \times 3.55$$
$$= 2513\ kN - m\ \underline{\text{Ans.}}$$

(b) Absolute maximum bending moment

The absolute maximum bending moment would occur under 160 *kN* load. By previous example, we can say that for the absolute maximum bending moment under 160 *kN* load, the load should occupy such a position on the beam that the centre of the beam is mid. way between the *C.G.* of the load and 160 *kN* load.

For this let us first find the *C.G.* of the load system. Let *C.G.* is at x. From 40 *kN* load. Then,

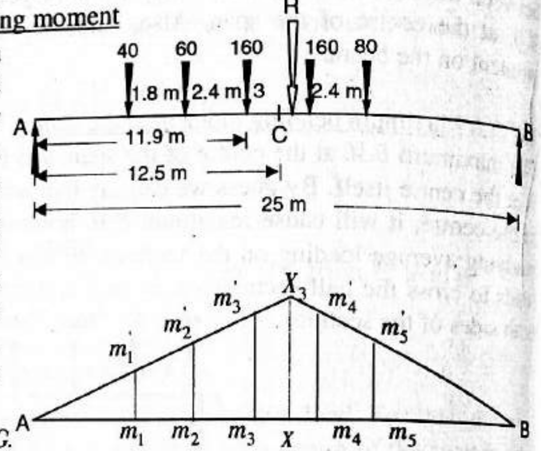


Fig. 5.27

$$\bar{x} = \frac{60 \times 1.8 + 160 \times 4.2 + 160 \times 7.2 + 80 \times 9.6}{40 + 60 + 160 + 80} = 5.4\ m$$

Therefore, 160 *kN* load should be at a distance

$$= 12.5 - \frac{1}{2}(5.4 - 4.2) = 11.9\ m \text{ from end } A. \text{ (Fig. 5.27)}$$

Now draw the influence line diagram for the *B.M.*

$$XX_3 = \frac{x(\ell - x)}{\ell} = \frac{12.5(25 - 12.5)}{25} = 6.25$$

From ΔAX_3B, we find that

$$m_1 m_1' = 3.85,\quad m_2 m_2' = 4.75,\quad m_3 m_3' = 5.95,\quad m_4 m_4' = 5.05 \text{ and } m_5 m_5' = 3.85$$

Absolute maximum bending moment

$$M_{abs}.\max = 40 \times m_1 m_1' + 60 \times m_2 m_2' + 160 \times m_3 m_3' + 160 \times m_4 m_4' + 80 \times m_5 m_5'$$
$$= 40 \times 3.85 + 60 \times 4.75 + 160 \times 5.95 + 160 \times 5.05 + 80 \times 3.85$$
$$= 2520.2\ \text{kN} - \text{m}\ \underline{\text{Ans.}}$$

5.10 *ILD* FOR GIRDERS SUPPORTING FLOOR BEAMS

Beams of big depth, which are used in bridges, are called girders. In most cases, loads in girders are applied indirectly through a floor system over it. A typical example of such construction is shown in Fig. (5.28).

In the system, floor slabs transfer the loads to the secondary beams and the secondary beams transfer the loads to girders. In the common system, as the unit load moves directly over the beam, influence line ordinate changes from point to point. However in the system of girder under the floor system the loads are only applied at the nodes.

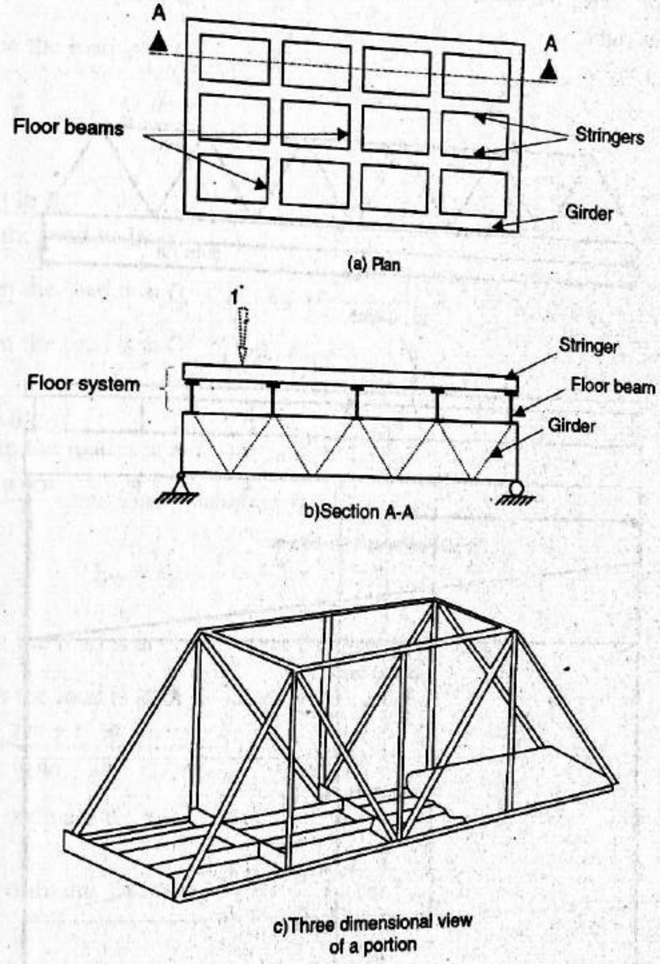


Fig. 5.28

In such application, *S.F.* and *B.M.* for a panel between the nodal points are constant. Therefore, influence lines for *S.F.* and *B.M.* are plotted for a panel and not for a particular section of a girder. Let us consider the girder system as shown in Fig. (5.29). The girder consists of n panels, each of length ℓ such that the total length $\ell = na$. The introduction of floor beams has no effect on the reactions R_A and R_B. Thus influence lines for R_A and R_B are similar to that of simply supported beams. The diagrams are shown in Fig. (5.29-c) and Fig. (5.29-d).

Let us draw an *ILD* for *S.F.* in a panel *CD*, which is $(m+1)^{th}$ panel from the left support. Then, there will be m panels to the left of *CD* and $(n-m-1)$ panels to its right.

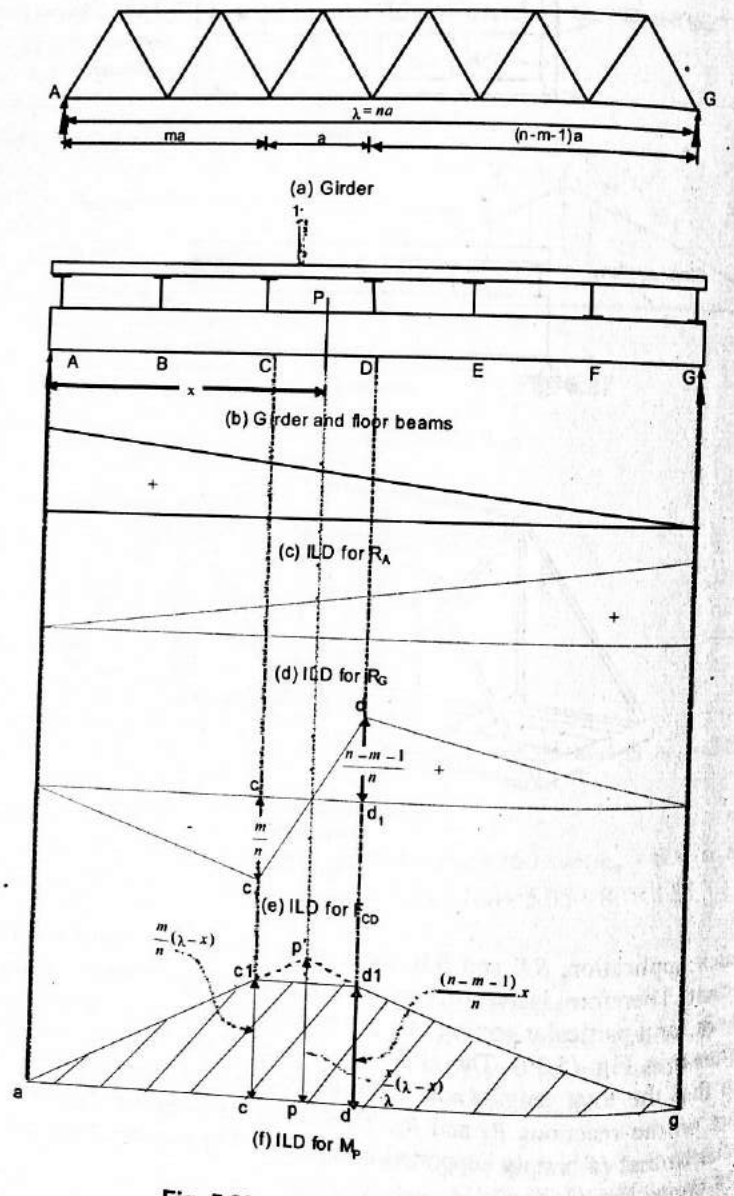


Fig. 5.29

(i) ILD for S.F. at P

(a) Load in *AC*

when the load is at *A*, $\quad R_G = 0, \quad F_{CD} = 0$

when the load is at *C*, $\quad R_G = \frac{ma}{na} = \frac{m}{n}$

$$F_{CD} = -R_G = -\frac{m}{n}$$

(b) Load in *DG*

Let the load be in *DG*,

when the load is at *D*, $\quad R_A = \frac{(n-m-1)a}{na} = \frac{n-m-1}{n}$

when the load is at G, $R_A = 0, \ F_{CD} = 0$

(c) Load in *CD*

When the load is at a distance x from *C*, load transferred at the point $C = \frac{a-x}{a}$ and load transferred at the point $D = -\frac{x}{a}$

$$\therefore \quad F_{CD} = R_G - \frac{x}{a} = -\frac{m}{n} + \frac{x}{a}$$

when the load is at *C*, $\quad x = 0, \ F_{CD} = -\frac{m}{n}$ as before

when the load is at *D*, $\quad x = a$

$$F_{CD} = \frac{ma+a}{na} - \frac{a}{a} = \frac{n-m-1}{n}$$

$\therefore$ ordinate $cc_1 = -\frac{m}{n}$

and ordinate $dd_1 = \frac{n-m-1}{n}$

(ii) ILD for B.M. at P

Let *P* is at a distance x from *A*

when the load is at *A*, $\quad M_p = R_G \times (\ell - x) = 0$ as $R_G = 0$

when the load is at *D*, $\quad M_p = R_A \times x$

$$\text{or, } dd_1 = \frac{n-m-1}{n} . x$$

when the load is at *C*, $\quad M_p = R_G(\ell - x)$

$$\text{or, } \quad cc_1 = \frac{m}{n} . (\ell - x)$$

The moments between C and D will remain constant due to the load being transferred through the panel points only. Therefore c_1d_1 can be joined. The ordinate pp' can be obtained by considering two similar triangles such that

$$\frac{pp'}{pa} = \frac{cc_1}{ca}$$

or, $$pp' = pa.\frac{cc_1}{ca}$$

or, $$pp' = x.\frac{m}{n}(\ell - x).\frac{1}{ma}$$

$$= \frac{x}{\ell}(\ell - x)$$

This is the ordinate for the girder without the floor beam. Thus alternatively, influence line for a girder with floor beams can be obtained by constructing the influence line for the beam assuming it to be without floor beam such that the maximum ordinate is $x\,(\ell - x)/\ell$ and joining p_1 to a and g as shown in Fig.(f). The *ILD* is thus represented by ac_1d_1g

5.11 Influence Lines for Plane Trusses

Trusses are used in long span bridges. They are mainly classified into the following two types depending upon the position of loading.

(i) Through Type: This type of truss receives load at its bottom chord joints as shown in Fig. (5.30).

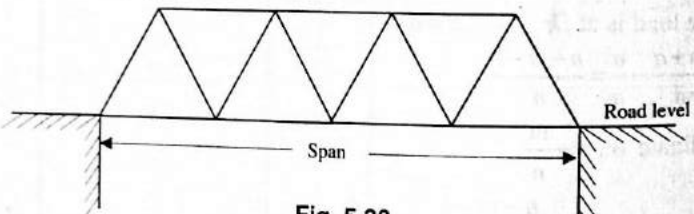


Fig. 5.30

(ii) Deck Type: This type of truss receives load at it as top chord joints as shown in Fig. (5.31)

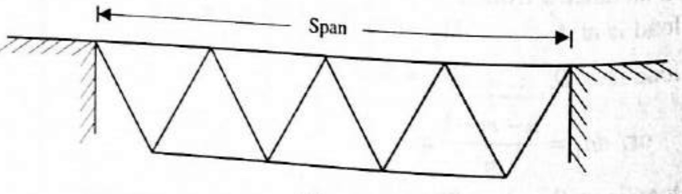


Fig. 5.31

(a) Influence line diagram for Pratt truss with parallel chords

ILD will be drawn for only the left half portion of the truss. The diagram of the other half will just be the repetition. For example, *ILD* for U_1U_2 and U_6U_7 are same.

Consider a Pratt truss shown in Fig.(5.32) and let the unit load moves at the bottom chord.

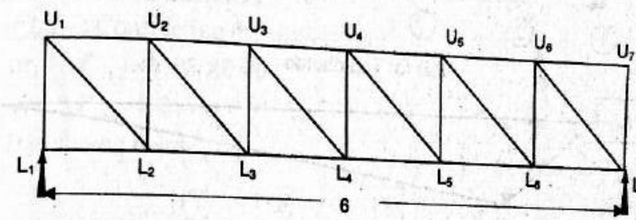


Fig. 5.32

We can visualize that under the application of load on the truss, all the top members experience compressive force while all the bottom members experience tensile force. But the vertical and inclined members experience tensile, compressive and both type of forces depending upon their location with respect to the applied load on the truss. The influence line diagram for the top and bottom members are given below with necessary explanations.

ILD for Top members

For U_1U_2

Let R_1 and R_7 are the reaction of the truss for any position of the unit load as shown in Fig. (5.33). Let us cut the section (1)–(1) and take moment from the joint L_2 (opposite of U_1U_2), then

$$F_{U1U2} \times h = R_1 \times a$$ (or alternatively, $F_{U1U2} \times h = R_7 \times 5a$, which will lead to the same result as R7=1/6)

or, $$F_{U1U2} = \frac{R_1 \times a}{h} = \frac{M_{L2}}{h} \qquad (\because R_1 \times a = \text{Moment at } L_2)$$

Thus it is seen that the influence line for force in member U_1U_2 is equal to $1/h$ times the influence line for M_{L2}. As from previous sections, influence line for M_{L2} is a triangle whose ordinate is equal to $x(\ell - x)/\ell = a(6a - a)/6a = 5a/6$ It is thus obvious that the influence line for the force in member U_1U_2 is also a triangle with ordinate equal to $5a/6h$ under the joint L_2 as shown by the Fig. (5.33).

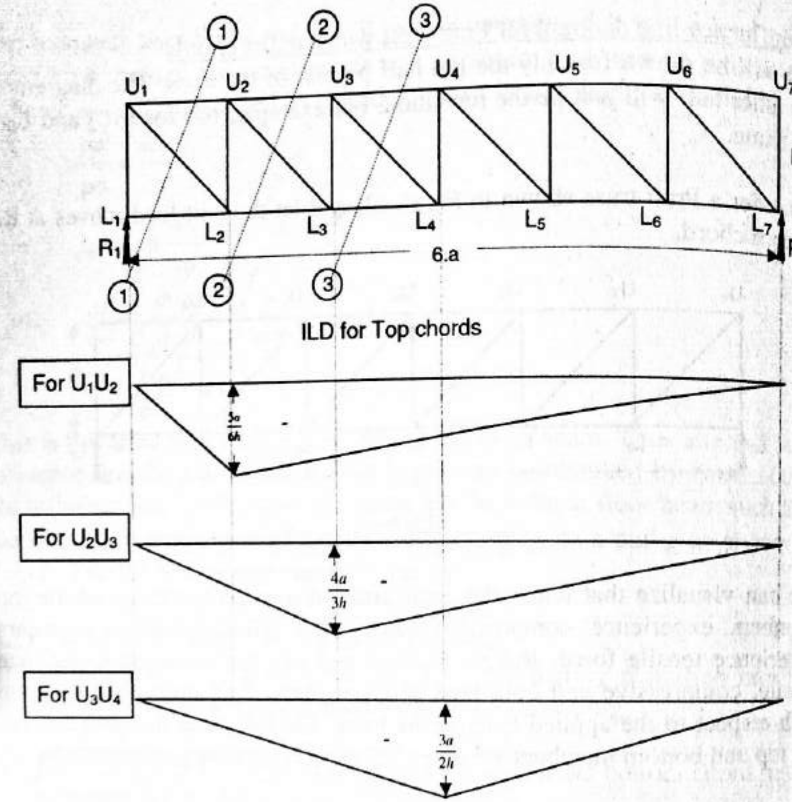


Fig. 5.33

For U_2U_3

Let us cut the section (2)–(2) and take moment about L_3,

$$F_{U2U3} \times h = R_1 \times 2a = M_{L3}$$

$$\therefore \quad F_{U2U3} = \frac{M_{L3}}{h}$$

Max. ordinate of *ILD* for ML_3 is obtained as $\dfrac{x(\ell - x)}{\ell} = \dfrac{2a(6a-2a)}{6a} = \dfrac{4a}{3}$

$\therefore$ Maximum ordinate of *ILD* for F_{U2U3} is $\dfrac{4a}{3h}$.

ILD drawn with the ordinate is shown in Fig. (5.33).

For U_3U_4

Pass the section (3)–(3), then $\sum M_{L4} = 0$

or, $\quad F_{U3U4} \times h = R_1 \times 3a = M_{L4}$

or, $\quad F_{U3U4} = \dfrac{M_{L4}}{h}$

Max. ordinate for *ILD* for M_{L4} is given by

$$\frac{x(\ell - x)}{\ell} = \frac{3a(6a-3a)}{6a} = \frac{3a}{2}$$

$\therefore \quad$ ordinate for $F_{U3U4} = \dfrac{3a}{2h}$ which is drawn in Fig. (5.33).

***ILD* for Bottom members**

For L_1L_2: Considering the section (1)–(1), $\sum M_{U1} = 0$

or, $\quad F_{L1L2} \times h = R_1 \times 0 = 0$

$\therefore \quad F_{L1L2} = 0$

ILD is shown below.

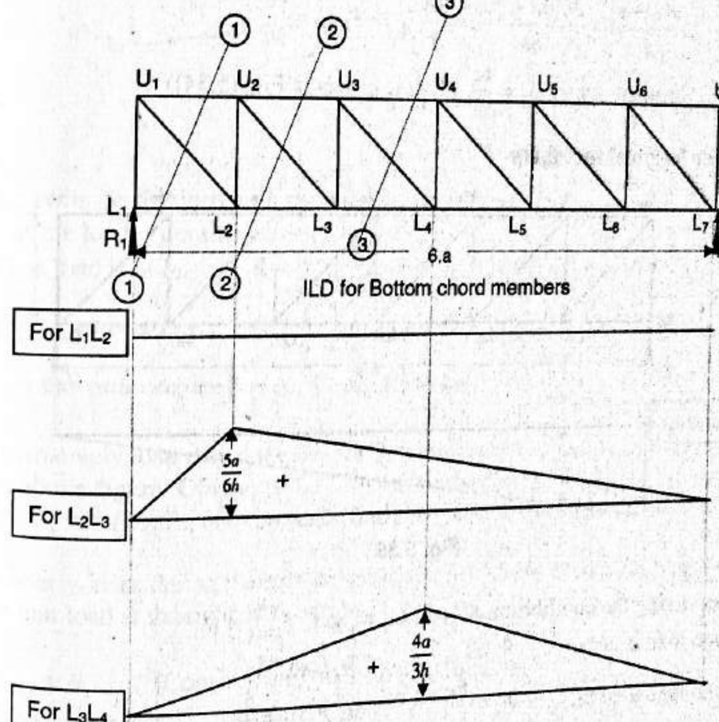


Fig. 5.34

For L_2L_3: Considering the section (2)–(2), $\sum M_{U2} = 0$

$$F_{L2L3} \times h = R_1 \times a = M_{L2}$$

or, $\quad F_{L2L3} \mp \dfrac{M_{L2}}{h}$

Max. ordinate for *ILD* for M_{L2} is given by

$$\frac{x(\ell-x)}{\ell}=\frac{a(6a-a)}{6a}=\frac{5a}{6}$$

$\therefore \quad F_{L2L3}=\dfrac{5a}{6h}$

$\therefore \quad$ ordinate for $F_{L2L3}=\dfrac{5a}{6h}$ (This is drawn in Fig. (5.34))

For L_3L_4: Considering the section (3)–(3), $\sum M_{U3}=0$

or, $\quad F_{L3L4}\times 4=R_1\times 2a$

or, $\quad F_{L3L4}=\dfrac{M_{L3}}{h}$

Max. ordinate of *ILD* for M_{L3} is given by

$$\frac{x(\ell-x)}{\ell}=\frac{2a(6a-2a)}{6a}=\frac{4a}{3} \quad \therefore F_{L3L4}=\frac{4a}{3h}$$

$\therefore \quad$ ordinate for $F_{U3U4}=\dfrac{4a}{3h}$ (This is drawn in Fig. (5.34))

***ILD* for Vertical members**

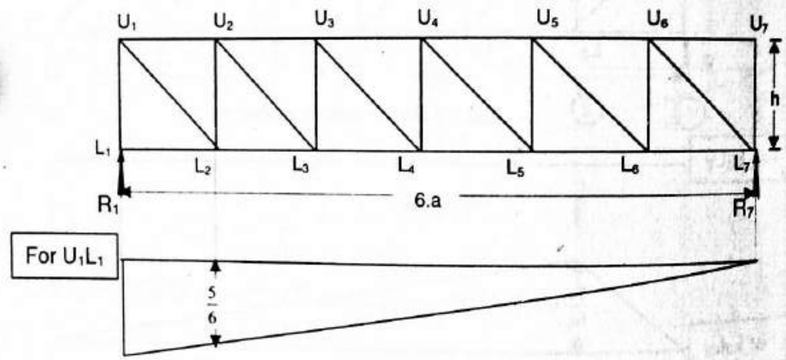


Fig. 5.35

For U_1L_1

Considering the equilibrium of joint L_1, we get, $F_{L1U1}=R_1$

When load is at L_1, $\quad R_1=1$, $\quad$ or, $F_{L1U1}=1$

When load is at L_2, $\quad R_1=\dfrac{5}{6}$, $\quad$ or, $F_{L1U1}=\dfrac{5}{6}$

When load is at L_7, $\quad R_1=0$, $\quad$ or, $F_{L1U1}=0$

The influence line drawn with the ordinates is shown in Fig. (5.35).

For U_2L_2

Referring Fig. (5.36), consider the equilibrium of right part of the section and assume that the unit load is at the left of L_2, then $F_{U2L2}=R_7$ (Tensile)

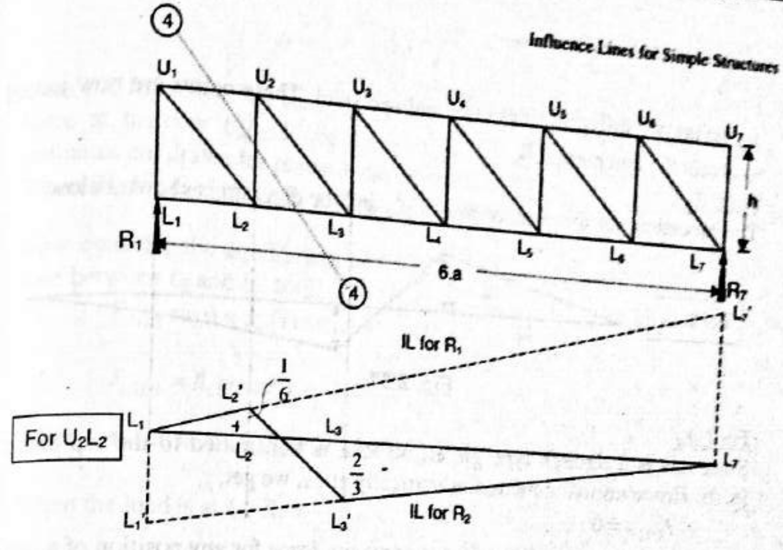


Fig. 5.36

The ordinate obtained with the above relation is valid only for points L_1 and L_2 as the load is considered at the left of L_2.

When load is at L_1, $\quad R_7=0 \quad \therefore$ ordinate at $L_1=0$

When load is at L_2, $\quad R_7=\dfrac{1}{6} \quad \therefore$ ordinate at $L_2=\dfrac{1}{6}$

With this ordinate, the line L_1, L_2' can be drawn.

Alternatively, one may draw *ILD* for R_7 with continuous and dotted line as in the above figure. Obviously $L_7L'_7=1$ and considering two similar triangles $L_1L_2L'_2$ and $L_1L_7L'_7$, one can easily find $L_2L'_2=1/6$. Now draw the line L_1L_2'.

Similarly, considering the equilibrium of left part of the section and assuming the unit load at the right of L_2, we get,

$F_{U2L2}=R_1$ $\quad$ (Compression) for load position between L_3 and L_7.

When load is at L_7, $\quad R_1=0$

When load is at L_3, $\quad R_1=\dfrac{4}{6}=\dfrac{2}{3}$

Now, draw the line L_7L_3'.

Thus it is clear that, when unit load moves from L_2 to L_3, the nature of force in the member changes from tension to compression as the two points L_2' and

L_3' lie on two different sides (+ve and –ve side). These points are now joined to obtain the complete *ILD*.

For U_3L_3

The procedure for this is same as of U_2L_2 and the diagram is shown below.

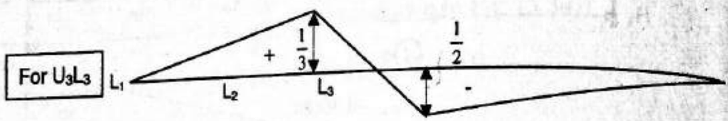


Fig. 5.37

For U_4L_4

Since this is a through type girder, no load is transmitted to the top chord joints. Hence resolving the forces vertically at U_4, we get,

$$F_{U4L4} = 0$$

Therefore, the member U_4L_4 will not carry any force for any position of a unit load on the bottom chord.

***ILD* for diagonal members**

For U_1L_2

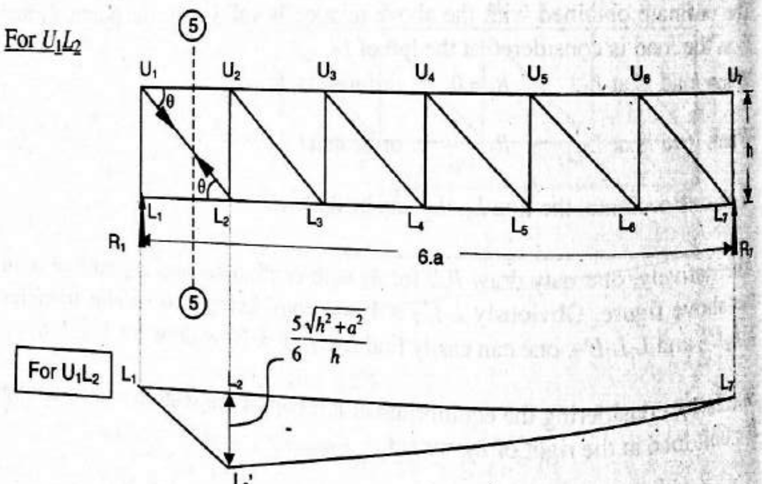


Fig. 5.38

Pass the section (5)–(5) and consider equilibrium of right portion assuming the unit load at L_1. Then,

$$F_{U1L2} \sin\theta = R_7$$

or, $\quad F_{U1L2} = R_7 \operatorname{cosec}\theta \qquad$ (Compressive force)

$$\text{But } \operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{\dfrac{h}{\sqrt{h^2+a^2}}} = \frac{\sqrt{h^2+a^2}}{h}$$

Thus the influence line for R_7 multiplied by cosec θ gives influence line for force at member U_1L_2. Since the load is considered at the left of L_2, the ordinates are drawn for points L_1 and L_2 only.

When the load is at L_1, $\; R_7 = 0$ i.e. $F_{U1L2} = 0$

Now consider the equilibrium of left portion assuming the position of unit load between L_2 and L_7, then,

$$F_{U1L2} \sin\theta = R_1 \text{ (Tensile)}$$

$$F_{U1L2} = R_1 \operatorname{cosec}\theta, \qquad \operatorname{cosec}\theta = \sqrt{\frac{h^2+a^2}{h}}$$

When the load is at L_7, $R_1 = 0$, i.e. $F_{U1L2} = 0$,

When the load is at L_2, $R_1 = \dfrac{5}{6}$, $\;\therefore\; F_{U1L2} = \dfrac{5}{6}\operatorname{cosec}\theta = \dfrac{5}{6}\dfrac{\sqrt{h^2+a^2}}{h}$

ILD is now shown in Fig. (5.38).

For U_2L_3

Pass the section (6)-(6) as shown in Fig. (5.39) and consider equilibrium of right portion assuming the position of unit load between L_1 and L_2

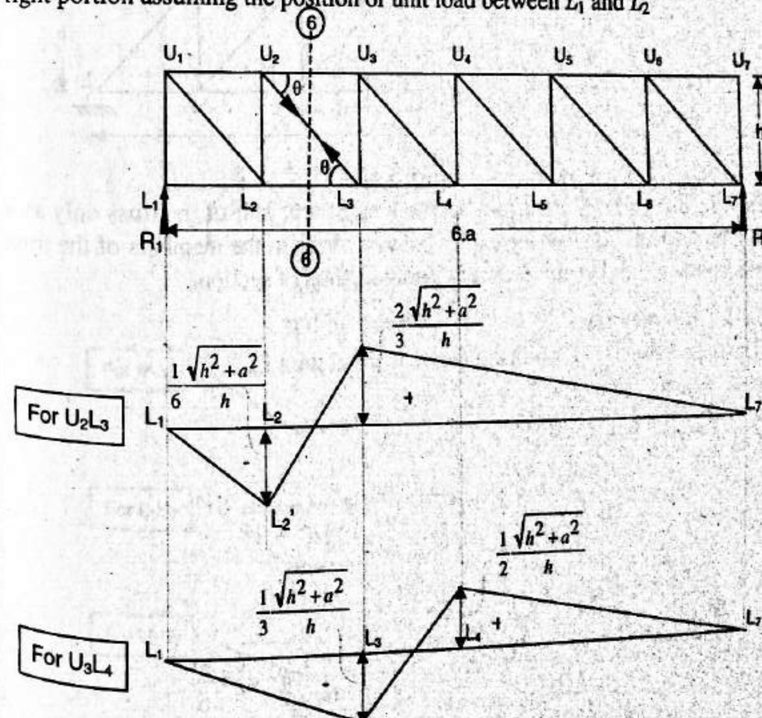


Fig. 5.39

$F_{U2L3} \sin\theta = R_7$ (Comp.) or $F_{U2L3} = R_7 \operatorname{cosec}\theta \;$ but, $\operatorname{cosec}\theta = \sqrt{\dfrac{h^2+a^2}{h}}$

When load is at L_1, $\quad R_7 = 0$, i.e $\quad F_{U1L2} = 0$

When load is at L_2, $\quad R_7 = \frac{1}{6} \quad \therefore \quad F_{U1L2} = \frac{1}{6}\frac{\sqrt{h^2+a^2}}{h}$

Similarly consider the equilibrium of left portion of the truss assuming the position of unit load between L_3 and L_7.

$F_{U2L3} \sin\theta = R_1$ (Tensile) $\quad$ or, $F_{U2L3} = R_1 \frac{\sqrt{h^2+a^2}}{h}$

When load is at L_7, $\quad R_1 = 0$, i.e. $F_{U2L3} = R_1$

When load is at L_3, $\quad R_1 = \frac{4a}{6a} = \frac{2}{3} \quad \therefore \ F_{U2L3} = \frac{2}{3}\frac{\sqrt{h^2+a^2}}{h}$

Now, the points of L_3 and L_2 are joined. The line goes from negative to positive side, which shows the change of nature of force.
In the same way *ILD* for F_{U3L4} can be drawn and it is shown in Fig. (5.39).

Example # 5.11 Draw influence line diagram for all the members of Pratt truss shown below.

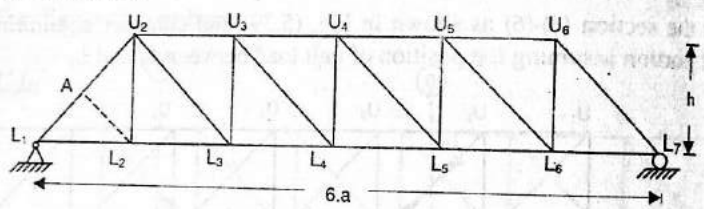


Fig. 5.40

Sol^n. Influence lines are drawn for the members of half of the truss only as it is symmetrical. For *ILD*, the expressions of forces in the members of the truss is necessary and they are drawn by using method of section.

***ILD* for top members**

Pass the section (1)-(1) and take moment about joint L_2,

$\Sigma M_{L2} = 0$

or, $\quad F_{L1U2} \times AL_2 = R_1.a$

or, $\quad F_{L1U2} = \frac{M_{L2}}{AL_2} \quad$ (by geometry, $AL_2 = \frac{a.h}{\sqrt{h^2+a^2}}$)

or, $\quad F_{L1U2} = M_{L2} \times \frac{\sqrt{h^2+a^2}}{h}$

But the maximum ordinate of $M_{L2} = \frac{x(\ell-x)}{\ell} = \frac{a}{6a}(6a-a) = \frac{5a}{6}$

$\therefore \quad F_{L1U2} = \frac{5a}{6} \times \frac{\sqrt{h^2+a^2}}{h}$

ILD with the ordinate is shown in the Fig.(5.41).

For U_2U_3

Pass the section (2)-(2) and take moment about joint L_3,

$\Sigma M_{L3} = 0$, $\quad$ or, $F_{U2U3} \times h = R_1 \times 2a$, $\quad \therefore F_{U2U3} = \frac{M_{L3}}{h}$

The maximum ordinate of M_{L3} in the *ILD* is given by

$$\frac{x(\ell-x)}{\ell} = \frac{2a(6a-2a)}{6a} = \frac{4a}{3}$$

Maximum ordinate of *ILD* for F_{U2U3} = 4a/3h, *ILD* is shown in the Fig.(5.41).

For F_{U3U4}

Pass the section (3)-(3) and take moment about joint L_4,

$\Sigma M_{L4} = 0 \quad$ or, $F_{U3U4} \times h = R_1 \times 3a$, $\quad \therefore F_{U3U4} = \frac{M_{L4}}{h}$

The maximum ordinate of M_{L4} in the *ILD* is given by

$$\frac{x(\ell-x)}{\ell} = \frac{3a(6a-3a)}{6a} = \frac{3a}{2}$$

$\therefore$ Maximum ordinate of *ILD* for $F_{U3U4} = \frac{3a}{2h}$, *ILD* is shown in the Fig.(5.41).

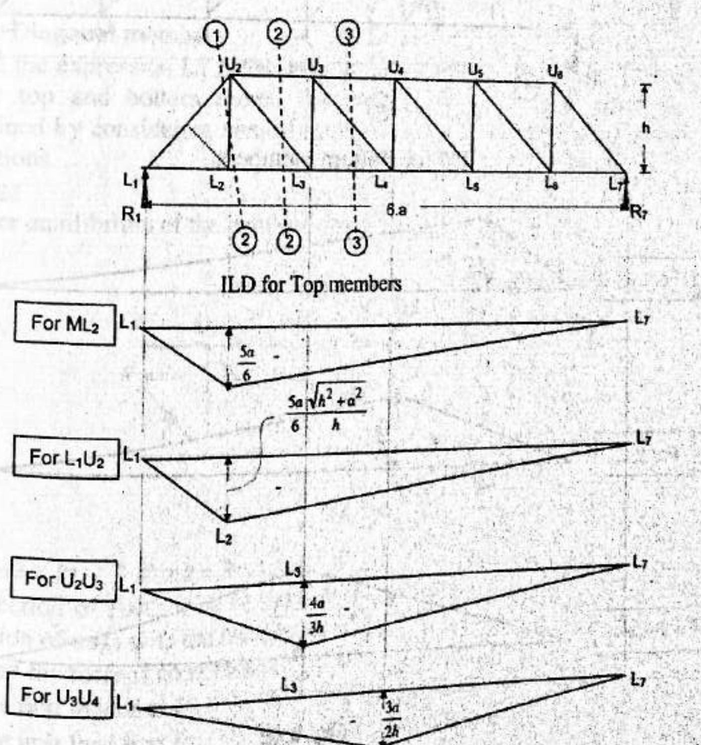


Fig. 5.41

The *ILD* for U_4U_5, U_5U_6 and U_6U_7 are same as that of U_3U_4, U_2U_3 and L_1U_2 respectively as from the symmetry of the diagram.

ILD for Bottom members

For F_{L1L2} Cutting the structure by sections (4)-(4) and taking moment about U_2, we get

$$F_{L1L2} \times h = R_1 \times a, \quad F_{L1L2} = \frac{R_1 \times a}{h} = \frac{M_{L2}}{h}$$

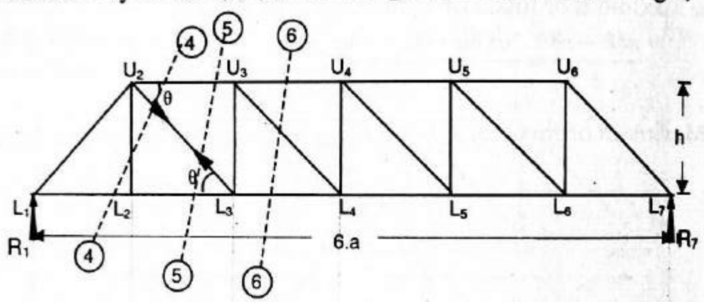
The maximum ordinate of M_{L2} in the *ILD* is given by

$$\frac{x}{\ell}(\ell - x) = \frac{a}{6a}(6a - a) = \frac{5a}{6}$$

$$\therefore \quad \text{For } F_{L1L2} = \frac{5a}{6h}$$

For F_{L2L3}

Cutting the structure by section (5)-(5) and taking moment about U_3, we get,



ILD for Bottom members

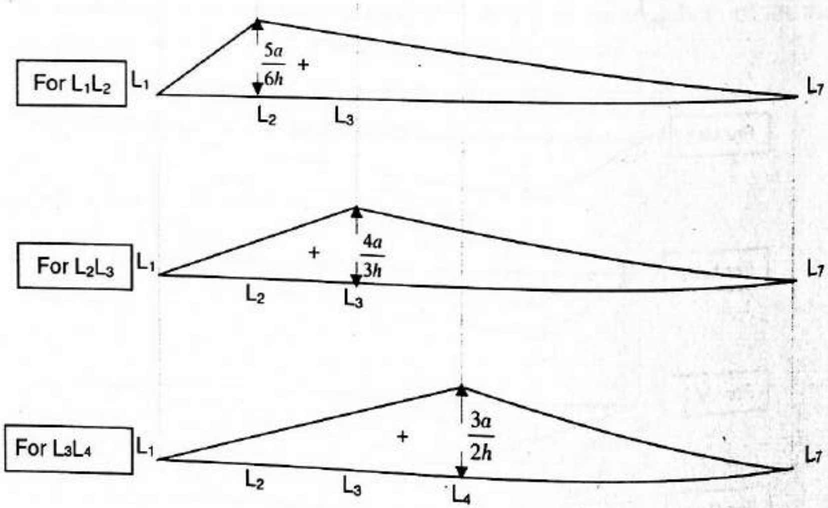


Fig. 5.42

$$F_{L2L3} \times h = R_1 \times 2a$$

$$F_{L2L3} = \frac{R_1 \times 2a}{h} = \frac{M_{L3}}{h}$$

The maximum ordinate of M_{L3} diagram is

$$\frac{x}{\ell}(\ell - x) = \frac{2a}{6a}(6a - 2a) = \frac{4a}{3}$$

$$\therefore \quad \text{For } F_{L2L3} = \frac{4a}{3h}$$

For F_{L3L4}

Passing the section (6)-(6) and $\sum M_{L4} = 0$

or, $\quad F_{L3L4} \times h = R_1 \times 3a$

or, $\quad F_{L3L4} = \dfrac{R_1 \times 3a}{h} = \dfrac{M_{L4}}{h}$

Maximum ordinate of $M_{L4} = \dfrac{3a}{6a}(6a - 3a) = \dfrac{3a}{2}$

$$\therefore \quad \text{For } F_{L3L4} = \frac{3a}{2h}$$

The diagrams are shown in Fig. (5.42)

ILD for Diagonal members

To find the expression for member forces, method of section is used. Unlike for the top and bottom chords, the expressions for member forces are determined by considering vertical equilibrium of the portion of truss cut by the sections.

For F_{U2L3}

Consider equilibrium of the right portion of the truss cut by section (7)-(7),

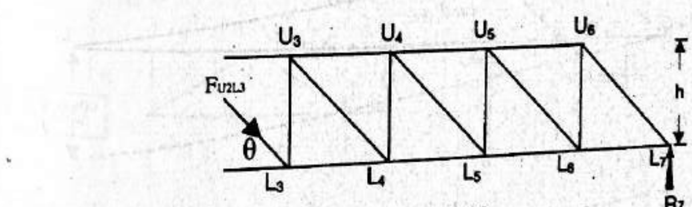


Fig. 5.43

$F_{U2L3} \sin\theta = R_7, \qquad F_{U2L3} = R_7 \operatorname{cosec}\theta$

The direction of F_{U2L3} must be towards the joint L_3 to keep equilibrium for the portion of truss with the reaction force R_7,

Therefore the force is compressive.

when the unit load is at L_1, $\quad R_7 = 0 \quad \therefore F_{U2L3} = 0$

when the unit load is at L_2, $\quad R_7 = 1/6 \quad \therefore F_{U2L3} = 1/6$

For the unit load position between L_3 and L_7, consider the equilibrium of left hand portion of the truss,

F_{U2L3} must be in tension as seen from the cut section of Fig. (5.44)

Now, $R_1 = F_{U2L3}\sin\theta$

Or, $F_{U2L3} = R_1 \operatorname{cosec}\theta$

when the unit load is at L_7,

$R_1 = 0$ $\quad\therefore F_{U2L3} = 0$

when the unit load is at F_3, $\quad F_1 = 2/3$

$\therefore F_{U2L3} = 2/3 \operatorname{cosec}\theta$

ILD is shown in Fig. (5.45)

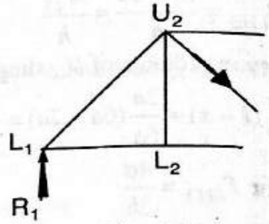


Fig. 5.44

For F_{U3L4}

ILD for the force in this member is drawn similar to that for F_{U2L3}. The truss is cut by the section (8)-(8) for it. The final *ILD* is shown in Fig. (5.45).

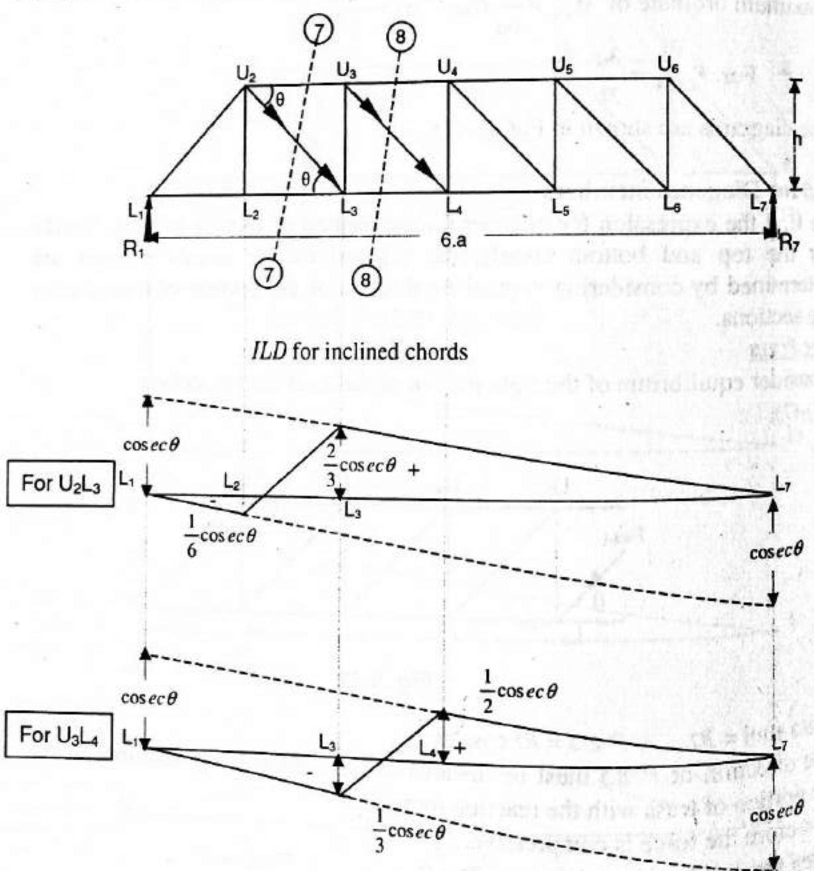


Fig. 5.45

ILD for Vertical members

For F_{U2L2}

The forces in the member U_2L_2 will be zero when the unit load is at L_1. The force increases as it moves towards L_2 and its magnitude will be 1 (equal to the applied load) when the unit load is over L_2. As it leave the point and move towards L_3, the force in the member gets decreased and its value will be zero when the load is directly over L_3. For any position of load right of L_3, the force in the member will be zero. *ILD* for U_2L_2 is shown in Fig. (5.46)

For F_{U3L3}

Consider the right portion of the section (9)-(9), and resolve the forces vertically, $\quad F_{U3L3} = R_7$ (Tensile)

When the unit load is at the right side of the joint L_4, consider the left part of the section (9)-(9). Resolving vertically, we have,

$$F_{U3L3} = R_1 \text{ (Compressive)}$$

The *ILD* for R_1 and R_7 are drawn.

The part of *ILD* for R_7 between L_1 and L_3 and the part of the *ILD* for R_1 between L_4 and L_7 will now be considered and the *ILD* for the member U_2L_3 is shown in Fig. (5.46).

For F_{U4L4}

As the load moves on the bottom chord, the force in the member U_4L_4 will always be zero. This is also evident if we cut a joint U_4 and consider the free body diagram. The influence line for the member will thus be a straight line coinciding with the base as shown in Fig. (5.46).

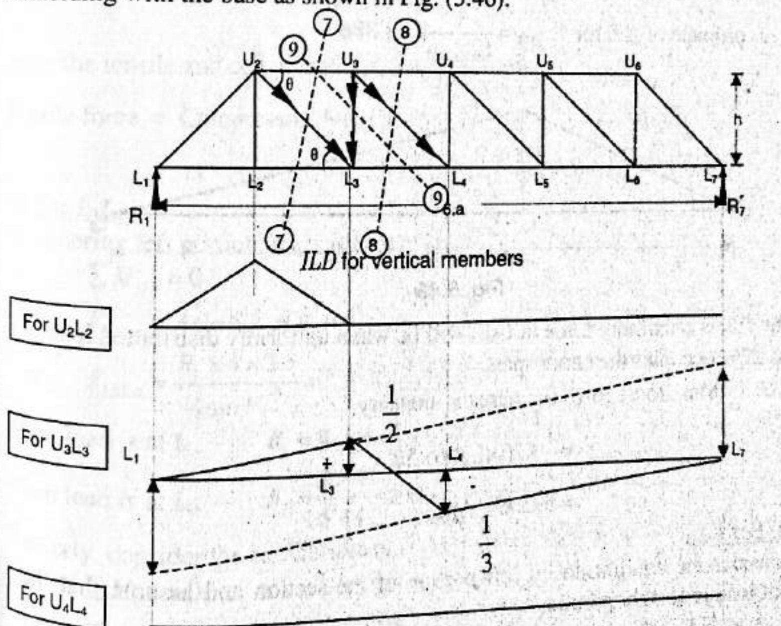


Fig. 5.46

Example # 5.12 Determine the maximum forces in the members U_2U_3, L_3U_3 and L_3L_4 of the bridge truss shown in Fig. (5.47). If a *udl* of 50 kN/m longer than the span traverses along the bottom of the chord members.

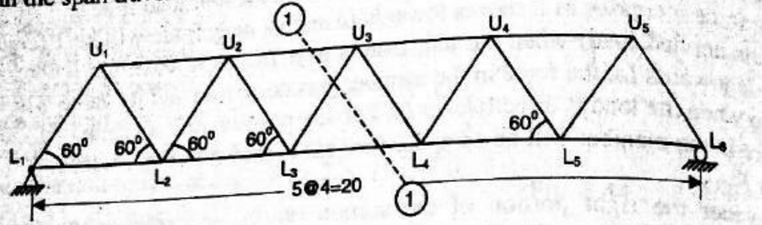


Fig. 5.47

Sol^n.

(a) For U_2U_3

Consider the left position of the section (1)-(1), then

$$\Sigma M_{L3} = 0$$

or, $\quad F_{U2U3} \times 4\sin 60^0 = R_1 \times 2\times 4$ (Compressive)

$$F_{U2U3} = R_1 \times \frac{2\times 4}{4\sin 60^0} = \frac{ML_3}{4\sin 60^0}$$

Max. ordinate of *ILD* for M_{L3} is given by

$$\frac{x(\ell - x)}{\ell} = \frac{2\times 4(20-8)}{20} = 4.8$$

$\therefore$ ordinate of *ILD* for $F_{U2U3} = \dfrac{4.8}{4\sin 60^0} = 1.386$

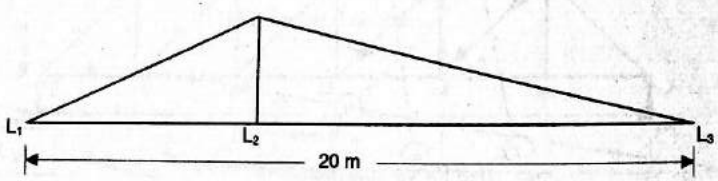


Fig. 5.48

The *ILD* is maximum force in U_2U_3 will be when uniformly distributed load of 50 kN/m occupies the entire span.

$\therefore \quad$ Max. force in U_2U_3 = area × intensity

$$= \frac{1}{2} \times 20 \times 1.386 \times 50$$

$$= 693\ kN \ \underline{\text{Ans.}}$$

(b) For L_3U_3

Consider the equilibrium of left portion of the section and assume that the unit load is at right portion.

$$\Sigma F_y = 0$$

or, $\quad F_{L3U3} \sin 60^0 = R_1$ (Compressive)

or, $\quad F_{L3U3} = R_1 \operatorname{cosec} 60^0$

When unit load is at L_6, $R_1 = 0$, $\quad$ i.e. $F_{L3U3} = 0$

When unit load is at L_4, $R_1 = \dfrac{2\times 4}{5\times 4} = \dfrac{2}{5} \quad \therefore \; F_{L3U3} = \dfrac{2}{5} \times \operatorname{cosec} 60^0 = 0.46$

Similarly consider the vertical equilibrium of right portion of the section (1)-(1) and assume the unit load is at left portion.

$$\Sigma F_y = 0$$

$$F_{L3U3} \sin 60^0 = R_6 \text{ (Tensile)}$$

or, $\quad F_{L3U3} = R_6 \operatorname{cosec} 60^0$

When unit load is at L_1, $R_6 = 0$ $\quad$ i.e. $F_{L3U3} = 0$

When unit load is at L_3, $R_6 = \dfrac{2\times 4}{5\times 4} = \dfrac{2}{5} \quad \therefore \; F_{L3U3} = \dfrac{2}{5} \operatorname{cosec} 60^0 = 0.46$

ILD is shown below.

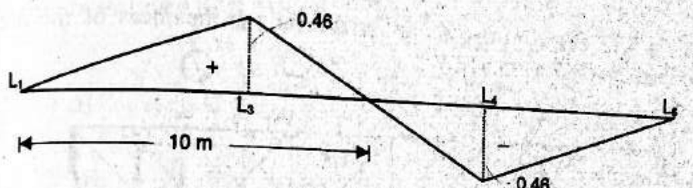


Fig. 5.49

Since the tensile and compressive ordinate of *ILD* is same,

Tensile force = Compressive force = area $\times w = \dfrac{1}{2} \times 10 \times 0.46 \times 50$

$$= 115\ kN \ \underline{\text{Ans.}}$$

(c) For L_3L_4

Considering left portion of section (1)-(1) in Fig. (5.47),

$$\Sigma M_{U3} = 0$$

or, $\quad F_{L3L4} \times 4\sin 60^0 = R_1 \times 4 \times 2.5$

or, $\quad F_{L3L4} = \dfrac{R_1 \times 4\times 2.5}{4\sin 60^0} = 2.87R_1$ (Tensile)

When load is at L_6, $\quad R_1 = 0$ i.e. $F_{L3L4} = 0$

When load is at L_4, $\quad R_1 = \dfrac{2\times 4}{5\times 4} = 0.4 \;\therefore\; F_{L3L4} = 2.87\times 0.4 = 1.15\ m$

Similarly, consider the equilibrium of right portion of the section,

$$\Sigma M_{U3} = 0$$

or, $\quad F_{L3L4} \times 4\sin 60^0 = R_6 \times 4\times 2.5 \quad \therefore F_{L3L4} = 2.87\,R_6$

When load is at L_1, $R_6 = 0 \therefore F_{L3L4} = 0$

When load is at L_3, $R_6 = \frac{2\times4}{5\times4} = \frac{2}{5}$, $\therefore F_{L3L4} = 2.87\times0.4 = 1.15\ m$

This influence line will be

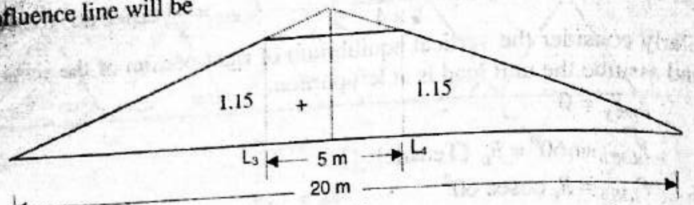


Fig. 5.50

$\therefore$ Max. force $F_{L3L4} = \left(\frac{5+20}{2}\right)\times1.15\times50$

$= 718.75\ kN$ Ans.

Example # 5.13 Draw influence line for the various members of the deck type girder shown in Fig. (5.51)

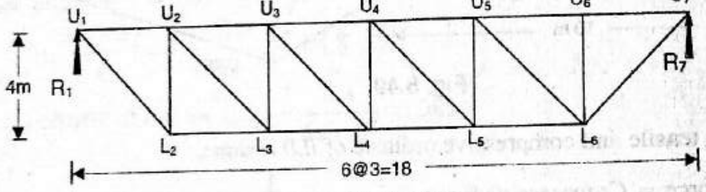


Fig. 5.51

Sol^n. As this is a deck type girder, the live load is transferred to the top chord joints.

For U_1U_2

Considering equilibrium of left part of section (1)–(1) in Fig. (5.52).

$\Sigma M_{L2} = 0$

$F_{U1U2}\times4 = R_1\times3$ (Compressive)

or, $F_{U1U2} = \frac{M_{L2}}{4}$

Maximum ordinate of $M_{L2} = \frac{x(\ell-x)}{\ell} = \frac{3(18-3)}{18} = 2.5$

Maximum ordinate of *ILD* for $F_{U1U2} = \frac{2.5}{4} = 0.625$

For U_2U_3

Considering left part of section (2)–(2)

$\Sigma M_{L3} = 0$

$F_{U2U3}\times4 = R_1\times2\times3$ (compressive)

or, $F_{U2U3} = \frac{M_{L3}}{4} = 0.25\ M_{L2}$

Maximum ordinate of $M_{L3} = \frac{x(\ell-x)}{\ell} = \frac{3\times2(18-6)}{18} = 4$

Maximum ordinate of *ILD* for $F_{U2U3} = 0.25\times4 = 1$

For U_3U_4

Considering left portion of section (3)–(3)

$\Sigma M_{L4} = 0$

$F_{U3U4}\times4 = R_1\times3\times3 = M_{L4}$ (Compressive)

$F_{U3U4}\times4 = 0.25\times M_{L4}$

Maximum ordinate of $M_{L4} = \frac{x(\ell-x)}{\ell} = \frac{3\times3(18-9)}{18} = 4.5\ m$

Maximum ordinate of *ILD* for $F_{U3U4} = 0.25\times4.5 = 1.125$

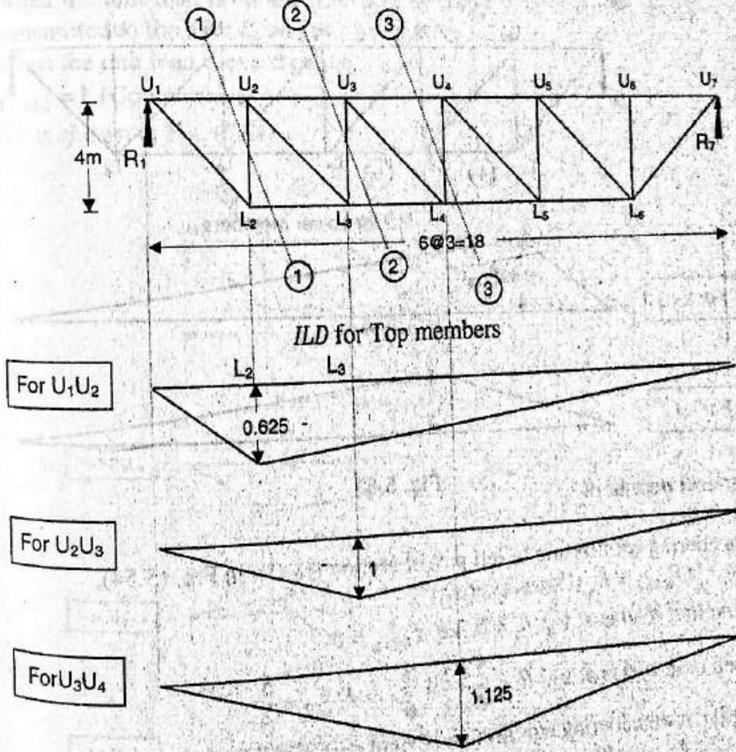


Fig. 5.52

Bottom members

For L_2L_3

Consider left part of section (4)–(4) in Fig. (5.53),

$$F_{L2L3} \times 4 = R_1 \times 3$$

or, $$F_{L2L3} = 0.25 \times M_{U2}$$

Maximum of $M_{U2} = \dfrac{x(\ell - x)}{\ell} = \dfrac{3(18-3)}{18} = 2.5\ m$

Maximum ordinate of *ILD* for $F_{L2L3} = 0.25 \times 2.5 = 0.625$

For L_3L_4

Consider left part of section (5)–(5)

$$F_{L3L4} \times 4 = R_1 \times 3 \times 2 = M_{U3}$$

or, $$F_{L2L3} = 0.25 \times M_{U3}$$

Maximum ordinate of $M_{U3} = \dfrac{x(\ell - x)}{\ell} = \dfrac{3 \times 2(18-6)}{18} = 4\ m$

Maximum ordinate of *ILD* for $F_{L3L4} = 0.25 \times 4 = 1\ m$

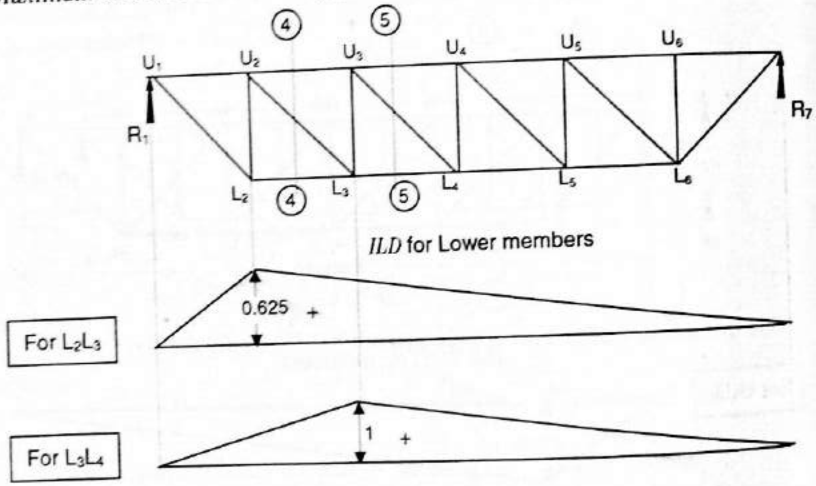


Fig. 5.53

Vertical members

For U_2L_2

Considering equilibrium of left part of section (1)–(1), in Fig. (5.54),

$$F_{U2L2} = R_1 \text{ (Compression)}$$

When unit load is at U_7, $R_1 = 0$, i.e. $F_{U2L2} = 0$

When unit load is at U_2, $R_1 = \dfrac{5 \times 3}{6 \times 3} = \dfrac{5}{6}$ $\therefore F_{U2L2} = \dfrac{5}{6}$

Similarly, considering equilibrium of right part of the section,

$$F_{U2L2} = R_7$$

When unit load is at U_1, $R_7 = 0$, i.e. $F_{U2L2} = 0$, *ILD* is drawn in Fig. (5.54).

For U_3L_3

Considering equilibrium of left part of the section (2)–(2) and the unit load at the right of it,

$$F_{U3L3} = R_1 \text{ (Compressive)}$$

When load is at U_7, $R_1 = 0$, i.e. $F_{U3L3} = 0$

When load is at U_3, $R_1 = \dfrac{4 \times 3}{6 \times 3} = \dfrac{2}{3}$ $\therefore F_{U3L3} = \dfrac{2}{3}$

Similarly, considering equilibrium of right portion of the section (2)–(2),

$$F_{U3L3} = R_7 \text{ (Tensile)}$$

When load is at U_1, $R_7 = 0$ i.e. $F_{U3L3} = 0$

When load is at U_2, $R_7 = \dfrac{1 \times 3}{6 \times 3} = \dfrac{1}{6}$ $\therefore F_{U3L3} = \dfrac{1}{6}$

Now, ends at U_2 and U_3 are jointed showing the change of nature of the force between these points.

For U_4L_4

When the unit load is on left side of U_3 or on the right side of U_5, no load is transmitted to the joint U_4 and will be no force in U_4L_4.

When the unit load is exactly at U_4,

$F_{U4L4} = 1$ (Compressive)

ILD is shown in Fig. (5.54).

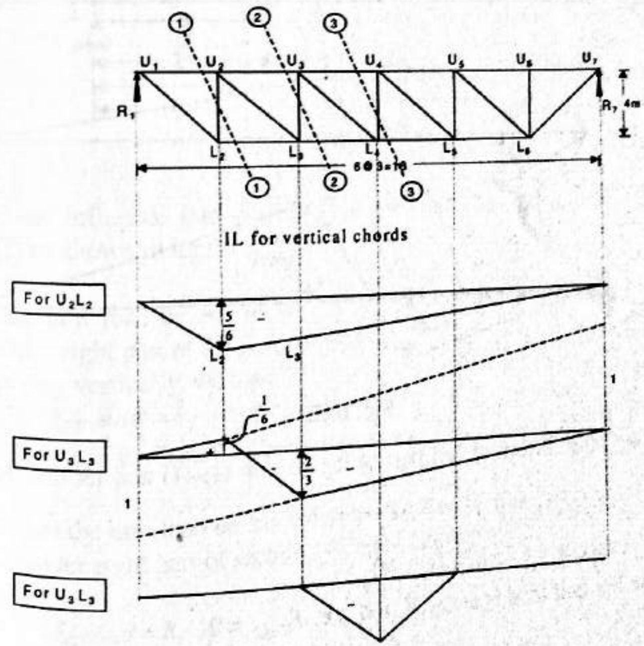


Fig. 5.54

(c) Diagonal members

For U_1L_2

When the load is at right side of U_2, resolving forces at joint U_1,

$\theta = \tan^{-1}\frac{4}{3} = 53.13^0$

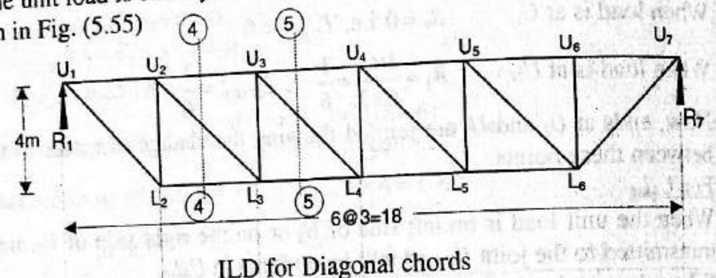
$F_{U1L2}\sin\theta = R_1$ (Tensile),

$\therefore\ F_{U1L2} = R_1 \operatorname{cosec}\theta = 1.25\,R_1$

When load is at U_7, $R_1 = 0$, i.e. $F_{U1L2} = 0$

When load is at U_2, $R_1 = \frac{5\times3}{6\times3} = \frac{5}{6}\ \therefore\ F_{U1L2} = 1.25\times\frac{5}{6} = 1.042$

When the unit load is exactly at U_1, there will be no force in U_1L_2 and the *ILD* is shown in Fig. (5.55)



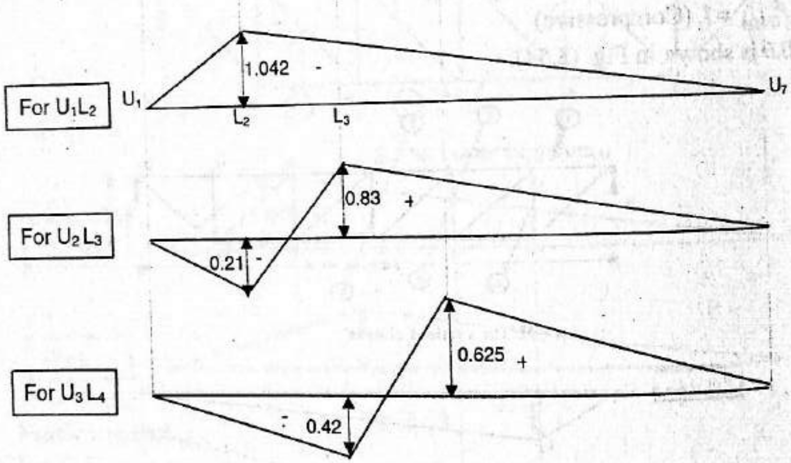


Fig. 5.55

FU_2L_3

Considering equilibrium of left part of the section (4)–(4) and let unit load is at right part,

Then, $F_{U2L3}\sin\theta = R_1$

$F_{U2L3} = R_1 \operatorname{cosec}\theta = 1.25\,R_1$ (Tensile)

When load the unit load is at U_7, $R_1 = 0$, i.e. $F_{U2L3} = 0$

When load the unit load is at U_3, $R_1 = \frac{4\times3}{6\times3} = \frac{2}{3}\ \therefore\ F_{U2L3} = 1.25\times\frac{2}{3} = 0.83$

Similarly, considering equilibrium of right part of the section.

$F_{U2L3}\sin\theta = R_7$

or, $F_{U2L3} = 1.25\times R_7$ (Compressive)

When the unit load is at U_1, $R_7 = 0$, i.e. $F_{U2L3} = 0$

When the unit load is at U_2, $R_7 = \frac{1\times3}{6\times3} = \frac{1}{6}\ \therefore\ F_{U2L3} = 1.25\times\frac{1}{6} = 0.21$

The ends U_2 and U_3 are jointed showing the change of nature of forces between these points.

For U_3L_4

ILD for this member is drawn exactly in the same way as above section (4)–(4) is to be considered. The diagram in shown in figure,

Example # 5.14 Draw influence line for the forces in bar *ED* of the truss shown in Fig. (5.56),

[T.U. 2055 Shrawan]

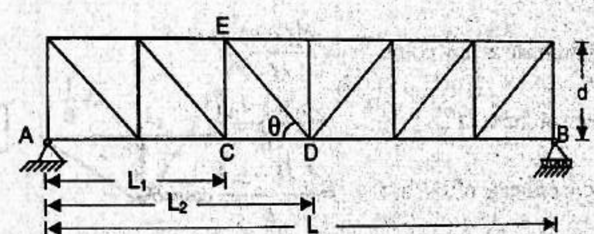


Fig. 5.56

Sol^n.

To draw influence line diagram for the forces on bar *ED*. Pass a section (1)–(1) as shown in figure.

Let the unit load be any where on the bottom chord on the left side of *D*. Consider right part of the section (1)–(1).

Resolving vertically, we have

$F_{ED}.\sin\theta = R_7$

$F_{ED} = R_7 \operatorname{Cosec}\theta$ (Compressive)

Now, Let the unit load be any where on the bottom chord on the right side of *D*. Consider right part of section 1–1 resolving vertically.

$F_{ED}\sin\theta = R_1$

$\therefore F_{ED} = R_1 \operatorname{Cosec}\theta$ (Tensile)

Now, the *ILD* for the quantity cosec θ R_1 is drawn. *ILD* for this will be a triangle whose altitude is cosec θ at the left end and zero at the right. Now the part of diagram on right side of L_2 only will be considered to represent the *ILD* for F_{ED} the range of movement of the unit load from *D* to right end of the lower members.

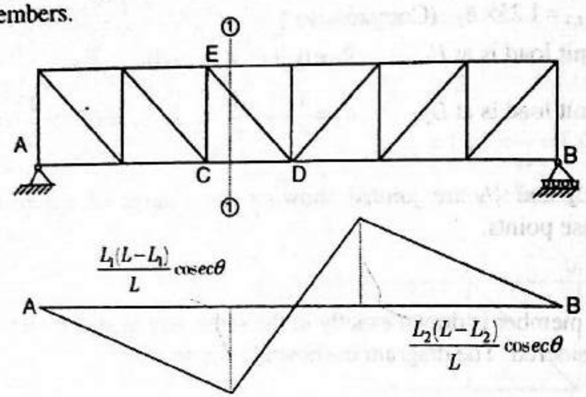


Fig. 5.57

Maximum ordinate at any point $= \dfrac{a(\ell - a)}{n\ell}$

∴ Maximum ordinate at *C* $= \dfrac{L_1(L-L_1)}{L}$

∴ Maximum ordinate of *ILD* at *C* $= \dfrac{L_1(L-L_1)}{L}\operatorname{cosec}\theta$

∴ Maximum ordinate at *D* $= \dfrac{L_2(L-L_2)}{L}$

Ordinate of *ILD* at *D* $= \dfrac{L_2(L-L_2)}{L}\operatorname{cosec}\theta$

Example # 5.15 Draw influence line diagram for the forces in the members of the through type bridge truss shown in Fig. (5.58).

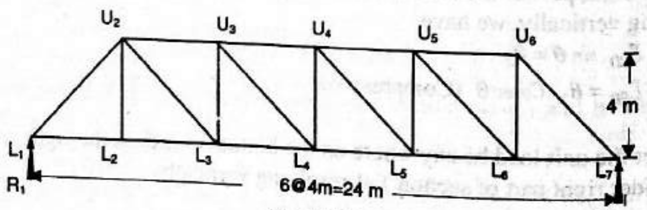


Fig. 5.58

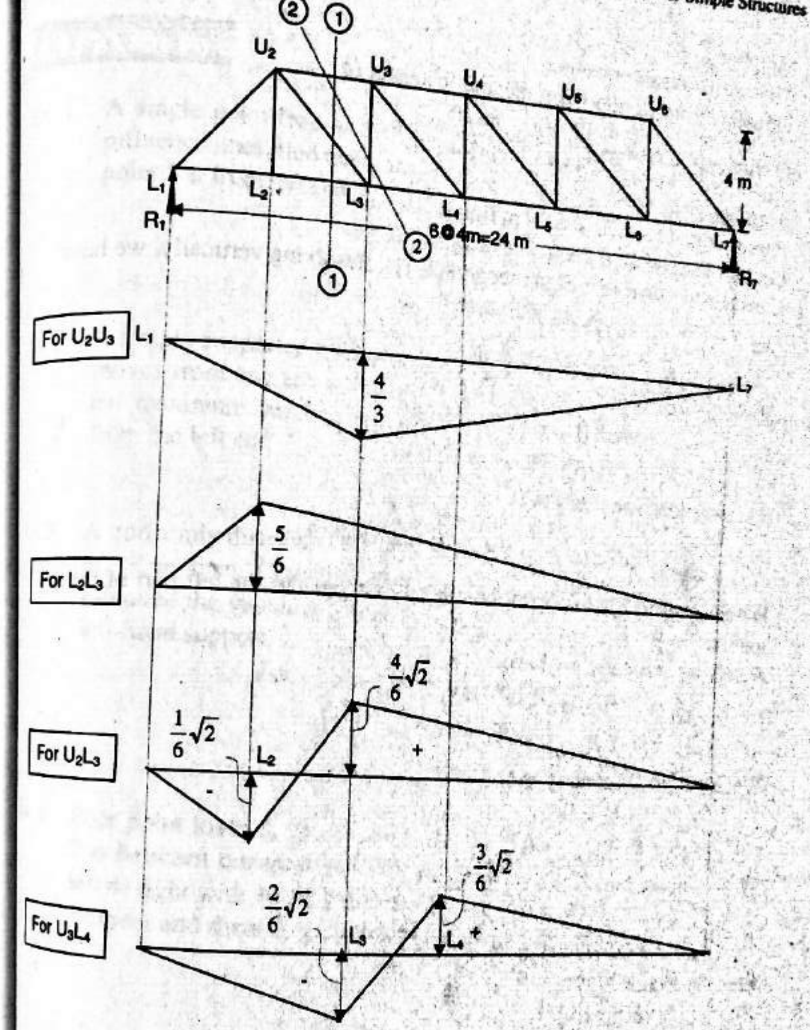


Fig. 5.59

Sol^n.

Top members

Member $\underline{U_2U_3}$

This is compression member

opposite joint is L_3.

Ordinate of *ILD* $= \dfrac{a\times(\ell - a)}{\ell h} = \dfrac{8\times(24-8)}{24\times 4} = \dfrac{8\times 16}{24\times 4} = 1.33\,m$

Now, the *ILD* for the quantity cosec θ R_1 is drawn. *ILD* for this will be a triangle whose altitude is cosec θ at the left end and zero at the right. Now the part of diagram on right side of l_2 only will be considered to represent the *ILD* for F_{ED} the range of movement of the unit load from *D* to right end of the lower members.

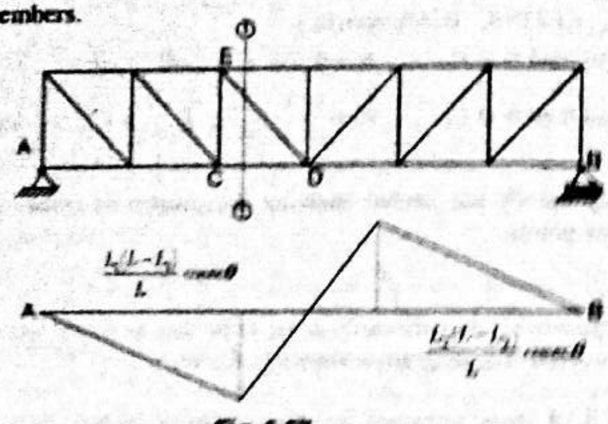


Fig. 5.57

Maximum ordinate at any point $= \dfrac{d(l-a)}{al}$

∴ Maximum ordinate at *C* $= \dfrac{l_1(l-l_1)}{l}$

∴ Maximum ordinate of *ILD* at *C* $= \dfrac{l_1(l-l_1)}{l}\,\text{cosec}\,\theta$

∴ Maximum ordinate at *D* $= \dfrac{l_2(l-l_2)}{l}$

Ordinate of *ILD* at *D* $= \dfrac{l_2(l-l_2)}{l}\,\text{cosec}\,\theta$

Example # 5.15 Draw influence line diagram for the forces in the members of the through type bridge truss shown in Fig. (5.58).

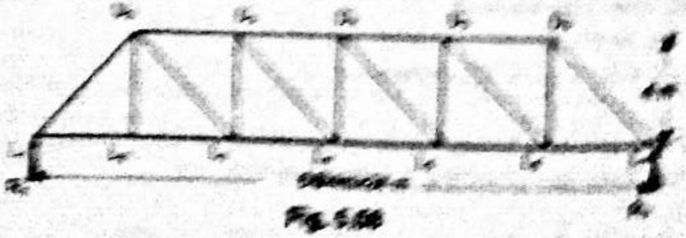


Fig. 5.58

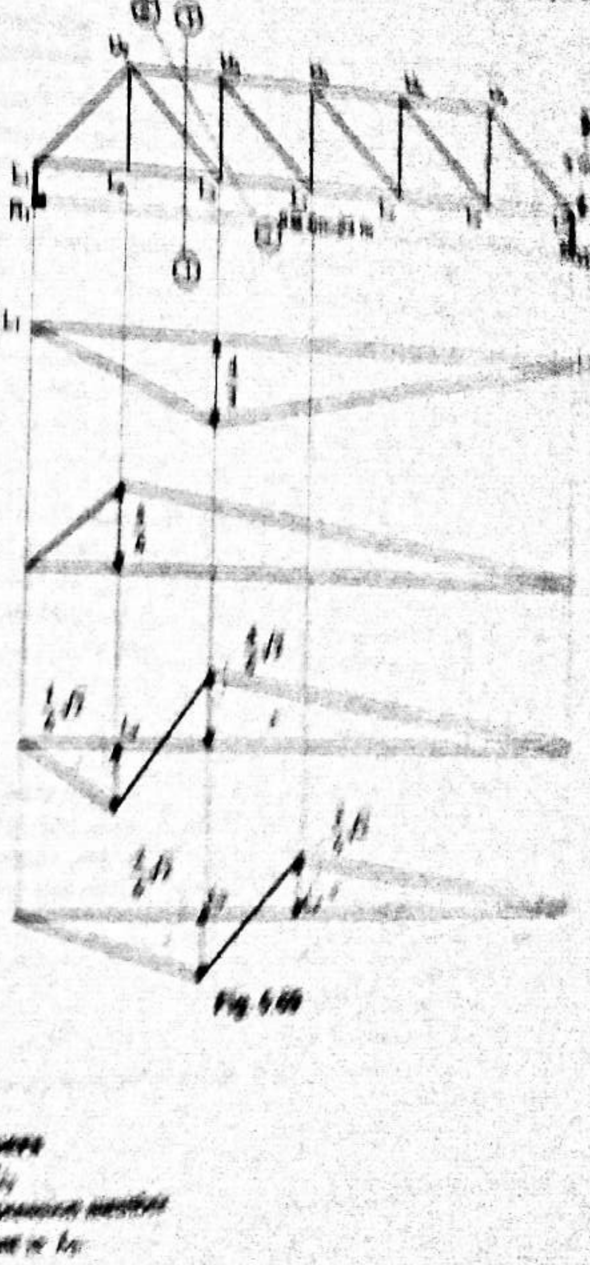


Fig. 5.59

Sol.

Member L_1L_2

This is the tension member

Opposite joint U_2 and when the unit load is at L_2

$$\text{Ordinate of ILD} = \frac{6\times(24-4)}{24\times 4} = \frac{6\times 20}{24\times 4} = \frac{5}{6}\,m$$

Member L_2L_3

Pass section (1)-(1) as shown in figure.

Let unit load be on the left side of L_2

Consider the right part of the section (1)-(1). Resolving vertically, we have

$$F_{U2L3}\sin 45^\circ = R_2$$

$$\therefore F_{U2L3} = \sqrt{2}\,.R_2$$

When the load is at L_1

$$R_2 = 0$$

$$\therefore F_{U2L3} = 0$$

When the load is exactly at L_2

$$F_{U2L3} = \frac{1}{6}\,\sqrt{2} \quad \text{(Compressive)}$$

When the unit load is on right side of L_3. Consider the left part of the section (1)-(1)

Resolving vertically, we have,

$$F_{U2L3}\sin 45^\circ = R_1$$

$$\therefore F_{U2L3} = \sqrt{2}\,.R_1$$

When the load is exactly at L_7

$$R_1 = 0$$

$$F_{U2L3} = 0$$

When the unit load is exactly at L_3

$$F_{U2L3} = \frac{4}{6}\,\sqrt{2} \quad \text{(Tensile)}$$

Pass the section (2)-(2)

Resolving vertically,

$$F_{U3L3} = R_1 \text{ (Tensile)}$$

$$= 2/6 = 1/3$$

When the unit load is anywhere on the right side of the joint L_4, consider left part of the section (1)-(1),

Resolving vertically, we have,

$$F_{U3L3} = R_1$$

$$= 3/6 = 1/2 \quad \text{(Compressive)}$$

6.12 Viva voce

Ex. 1 A single point load of 90 kN crosses a girder of 20 m span. Using influence lines, find maximum positive and negative S.F. and B.M. at a point 8 m from left end.

Ans. Max. Positive S.F. =[illegible] kN
Max. Negative S.F. =[illegible] kN
Maximum moment =[illegible] kNm

Ex. 2 A simply supported girder has a span of 12 m. A [illegible] tonne wheel load moves from one end to the other end on the span of the girder. Find the maximum bending moment which can occur at a section 6 m from the left end.

Ans. [illegible]

Ex. 3 A uniformly distributed load of 3 kN/m run, 4 m long crosses a girder of 16 m span. Construct the maximum S.F. and B.M. diagrams and calculate the values at the sections at the 3 m, 5 m and 8 m from the left-hand support.

Ans.

Section	Max. +ve S.F.	Max. −ve S.F.	Max. B.M.
3 m	[illegible] kN	[illegible]	[illegible]
5 m	[illegible] kN	[illegible]	[illegible]
8 m	[illegible] kN	[illegible]	[illegible]

Ex. 4 Four point loads 8, 15, 16 and 10 kN have center to center spacing of 2 m between consecutive loads traverse a girder of 30 m span from left to right with 10 kN load leading. Calculate the maximum bending moment and shear force at 8 m from the left support.

Max. +ve S.F.	Max. −ve S.F.	Max. B.M.
[illegible] kN	[illegible]	[illegible]

Ans.

6

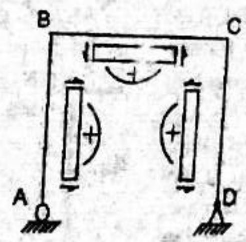
FRAMES AND ARCHES

6.1 THRUST, *BM* AND *SF* IN FRAMES

Frames in general can be defined as a structure made of several members forming a certain configuration. Frame with rigidly connected joints (rigid frames) can resist shear force, bending moment and axial forces. These forces can be exerts both at joints as well as along the members unlike in the case of pin-jointed frames where only the forces on joints are permitted. A typical rigid frame is shown in the figure below. The joints *B* and *C* are rigidly connected whereas the supports at *A* and *D* are hinge and roller respectively.

Steps for drawing moment shear and thrust diagrams.

1) Find the support reactions of the frame considering three equations of equilibrium of the entire structure. $\sum F_x = 0, \sum F_y = 0, \sum M = 0$.
2) Isolate each member from the frame and draw free body diagram for each of these.
3) Draw the diagrams taking the value of moments, shear and thrust from the free body diagram considering the following sign conventions. Bending moment is drawn at the tension side.



Sign convention

Fig. 6.1

Note that the moment, shear and axial force diagram for the frames can be drawn as that for beams. For appropriate sign convention, one has to view the structure from inside of the frame.

Example # 6.1 Draw moment, shear and thrust diagram for the rigid frame *ABCD* shown in Fig. (6.2)

Sol^n.

1) The reactions are determined considering the equilibrium of the whole structure.

$\sum F_x = 0,$

or, $R_{AH} - 20 = 0$

or, $R_{AH} = 20\ kN.$

$\sum M_A = 0,$

or, $R_D \times 8 - 60 \times 6 - 40 \times 2 - 20 \times 3 = 0$

$R_D = 62.5\ kN.$

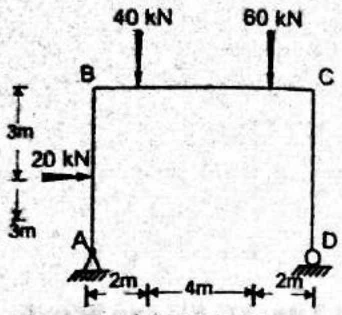
$\sum F_y = 0.$

or, $R_A + R_D - 40 - 60 = 0$

or, $R_A = 37.5\ kN.$

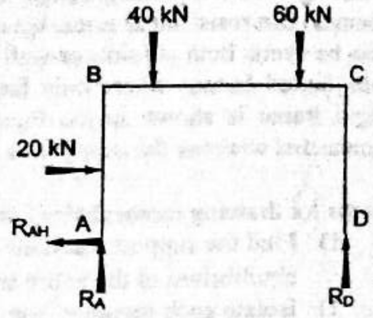
2) The free body diagram, of the frame is shown in Fig. (c).

Note: that when member AB is detached from the frame, one must add vertical force of $-37.5\ kN$ and the moment of $-20 \times 6 + 20 \times 3 = -60\ kNm$ at point B to keep it in equilibrium. The force and moment are balanced by the opposite forces added at point B of member BC such that sum of force and moment at joint B is equal to zero. It is clear that the horizontal forces (20-20=0) balances each other and it is not necessary to add any horizontal force at B in the member AB. Similarly one should add vertical force $37.5 - 40 - 60 = -62.5\ kN$ at point C to keep the point in equilibrium. The opposite of this force is applied at C of the member CD. The joint C of BC is horizontally in equilibrium as no horizontal force exists and also the moment at C is zero due to applied forces.

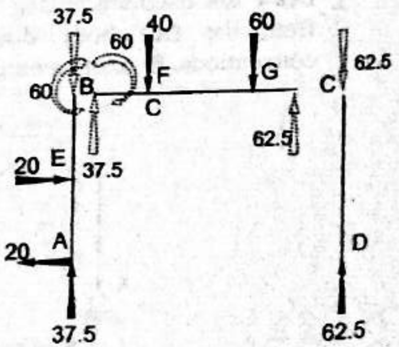


a) Given beam

Fig. 6.2



b) Reaction forces



c) Free body diagram

3) Now, bending moment, thrust and shear diagrams are drawn below.

(i) Member AB

For a section at x distance from B

at $x \le BE$,

$T_x = -37.5\ kN$ (Compression)

$F_x = 0, \qquad M_x = 60\ kN-m$ (Sagging)

at $x \ge BE$

$T_x = -37.5\ kN,$

$F_x = -20\ kN$ and $\quad M_x = 60 - 20(x-3)$

at $x = 3\ m, \qquad M_E = 60\ kN\text{-}m$

$x = 6\ m, \qquad M_A = 0$

(ii) For member BC,

$T_x = 0,$ (No axial force applied)

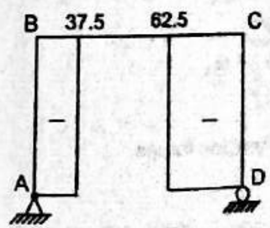
F_x at $B = -37.5$, at $F = -37.5 + 40 = +2.5\ kN$

F_x at $G = 2.5 + 60 = +62.5\ kN$

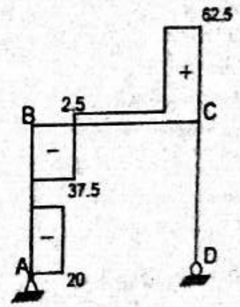
$M_B = 60\ kN\text{-}m$, $M_F = 60 + 37.5 \times 2 = 135\ kN\text{-}m$

$M_G = 60 + 37.5 \times 6 - 40 \times 4 = 125\ kN\text{-}m$

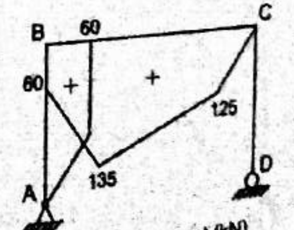
Final diagrams are:



d) Axial force (kN)

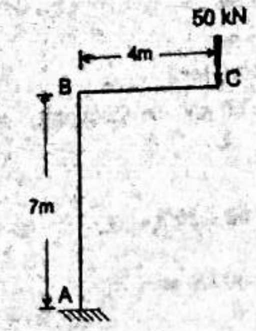


e) Axial force (kN)



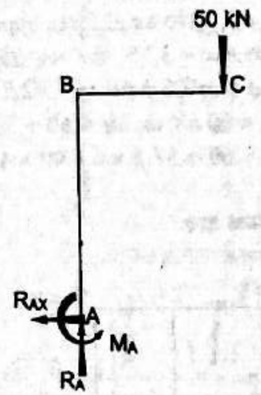
f) Bending moment (kN)

Example # 6.2 A rigid frame carries a concentrated load of 50 *kN* as shown in figure. Draw thrust, shear and bending moment diagrams for the frame.



a) Given beam

Fig. 6.3



b) Reaction forces

Sol^n

1) Finding reactions.

$\sum F_x = 0$

$R_{Ax} = 0$

$\sum F_y = 0$

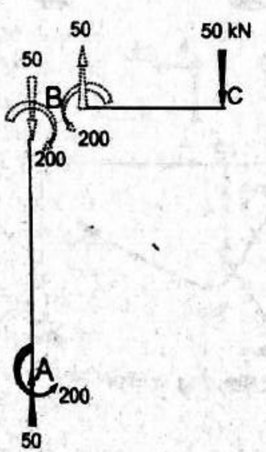
or, $R_A = 50$

$\sum M_A = 0$

$M_A - 50 \times 4 = 0$

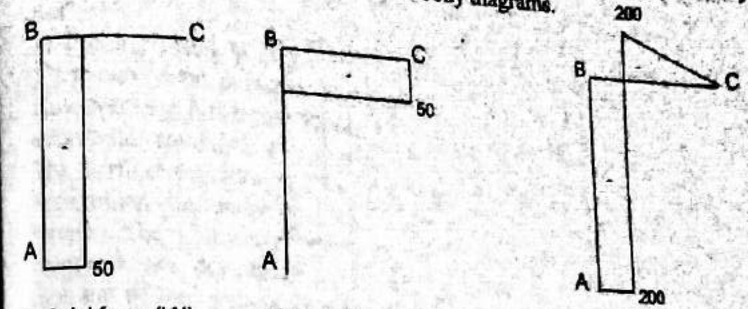
or, $M_A = 200$

2) Draw free body diagram,



c) Free body diagram

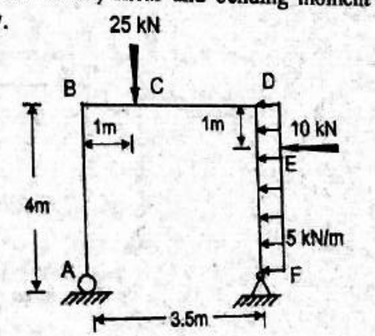
3) Thrust, moment and shear force diagrams can now be drawn by directly taking the values from the above free body diagrams.



d) Axial force (kN) e) Shear force (kN) f) Bending moment (kNm)

Note: In this problem, one may directly start to draw free body diagram without finding the reactions first. In the case, one has to start from point C.

Example # 6.3 Draw thrust, shear and bending moment diagrams for the frame shown below.

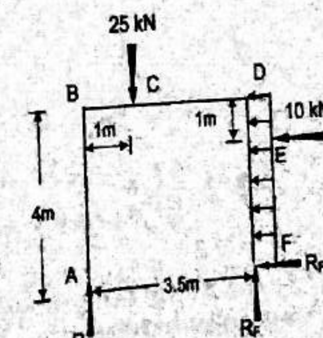


a) Given beam

Fig. 6.4

Sol^n

1) Finding reactions at supports:



b) Reaction forces

$\sum F_x = 0,$

$R_{Fx} - 10 - 5 \times 4 = 0$

or, $R_{Fx} = 30$

$\sum F_y = 0,$

or, $R_A + R_F = 25$ .... (i)

$\sum M_F = 0,$

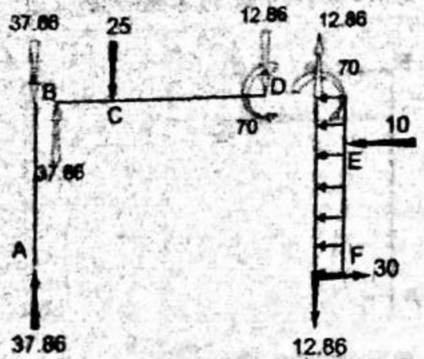
or, $R_A \times 3.5 + 25 \times 2.5 + 5 \times 4^2/2 + 10 \times 3$

or, $R_A = 37.86$

Substituting this value in Eq. (i)

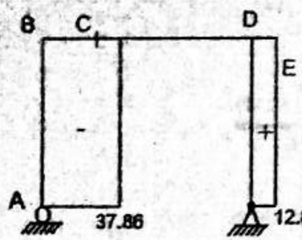
$R_F = -12.86$ (Direction is opposite to the assumed i.e. ↓)

2) Draw free body diagram.

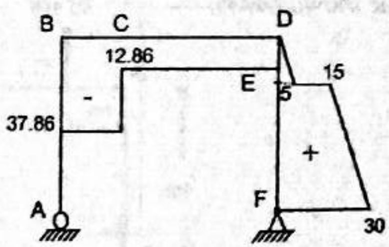


c) Free body diagram

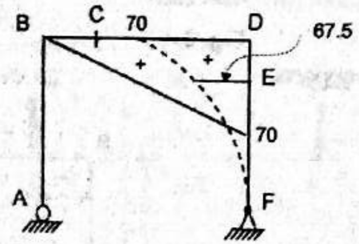
3) Draw the thrust, shear and bending moment diagrams.



d) Axial force (kN)



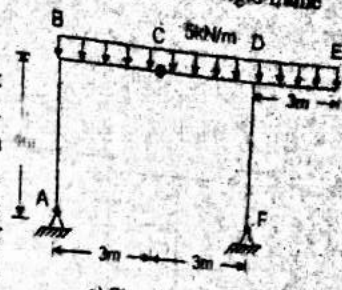
e) Shear force (kN)



f) Bending moment (kNm)

Note: Bending moment in the member varies in parabolic way due to the presence of uniformly distributed load. Moment at E which is $= 30 \times 30 - 5 \times 3^2/2 = 67.5$.

Example # 6.4 Draw thrust, shear and moment diagram of the rigid frame shown below.



a) Given beam

Fig. 6.5

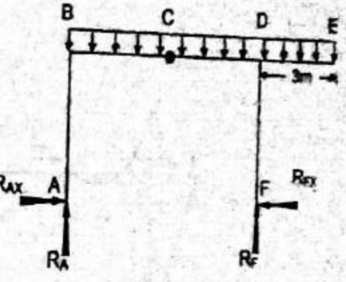
Sol^n

1) Finding reactions at the supports: There are four unknown reactions. However, the hinge at C provides an additional condition of equilibrium. The vertical reactions at A and F are determined considering equilibrium of whole the structure. Horizontal reactions are determined by taking moment of forces about C on one side of hinge only.

$\Sigma F_x = 0,\quad R_{Ax} - R_{Fx} = 0$.............(i)

$\Sigma F_y = 0,\quad R_A + R_F - 5 \times 9 = 0$

or, $R_A + R_F = 45$.....................(ii)



b) Reaction forces

$\Sigma M_A = 0,$

or, $R_F \times 6 - 5 \times \dfrac{9^2}{2} = 0$

or, $R_F = 33.75$

Substituting this value in Eq. (i) gives

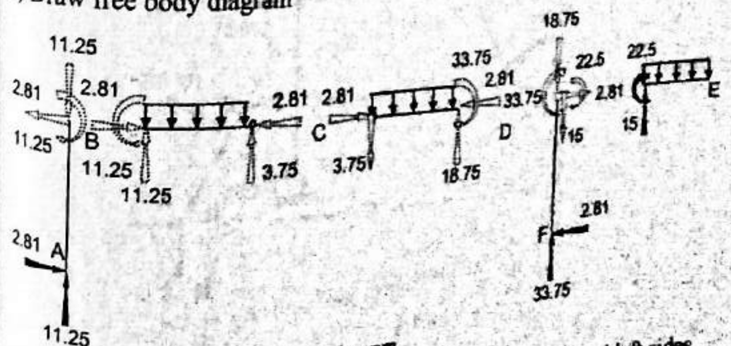
$R_A = 11.25$

Now, take moment about point C. Consider forces at the left of the hinge only, i.e. $\Sigma Mc = 0$, or, $-11.25 \times 3 + R_{Ax} \times 4 + 5\dfrac{3^2}{2} = 0$

or, $R_{Ax} = 2.81$

Substituting this value in Eq. (i), we get, $R_{Fx} = 2.81$

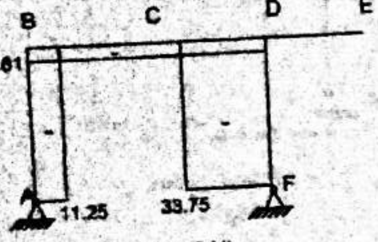
2) Draw free body diagram



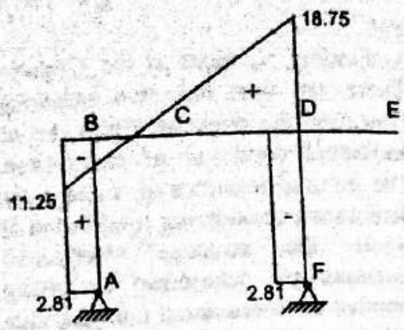
c) Free body diagram

Note: that at point D, balancing forces comes from both right and left sides.

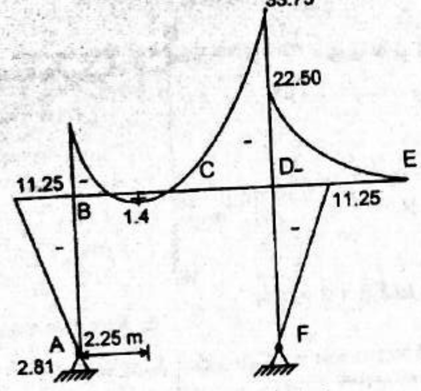
3) Draw the thrust, shear and moment diagrams as per the values obtained in the free body diagram.



d) Axial force (kN)

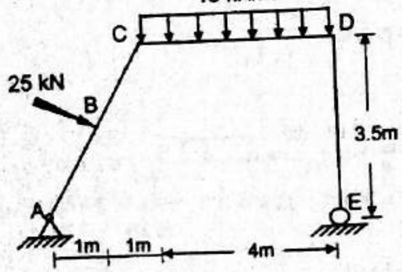


e) Shear force (kN)



f) Bending moment in (kNm)

Example # 6.5 Draw thrust, shear and bending moment diagrams for the frame shown below.

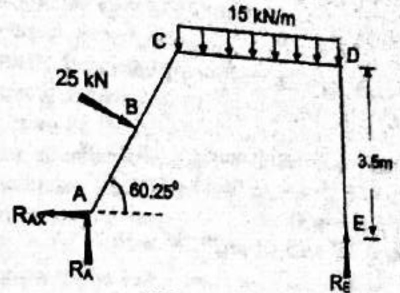


a) Given beam

Fig. 6.6

Sol^n

1) Find the reactions at supports



b) Reaction forces

Inclined distance

$$AB = \frac{1}{\cos 60.25^0}$$

$\sum F_x = 0$, or, $-R_A x + 25\cos 29.75^0 = 0$

or, $R_A x = 21.7$

$\sum F_y = 0$, or, $R_A + R_E - 25\sin 29.75^0 - 15 \times 4 = 0$

or, $R_A + R_E = 72.4$ .... (i)

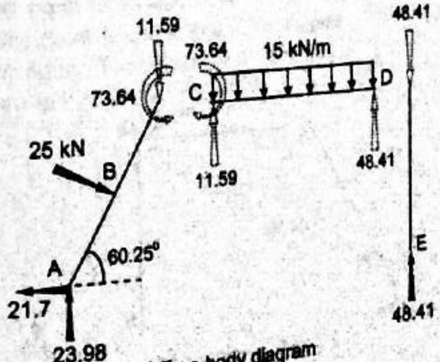
$\sum M_A = 0$, or, $R_E \times 6 - 15 \times 4 \times 4 - 25\sin 2.02 = 0$

or, $R_E = 48.41$

Substituting this value in Eq. (i)

$R_A = 23.99$

2) Draw free body diagram. It will be easier to start drawing free body diagram from the member *DE*.



c) Free body diagram

Unbalance moment at C for member CD

$$= 48.41 \times 4 - 15 \times \frac{4^2}{2} = 73.64\ kN-m$$

Balancing moment $= -73.64\ kN\text{-}m$

Now, moment at $B = 23.98 \times 1 + 21.7 \times \frac{3.5}{2} = 61.955$

3) Now draw thrust shear and banding moment diagrams.

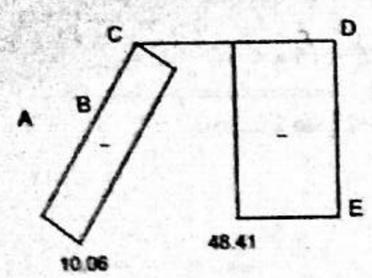
Thrust at $AC = 11.54 \cos 29.75^0 = -10.06\ kN$ (comp.)

$BC = 0$ and $DE = -48.41\ kN$

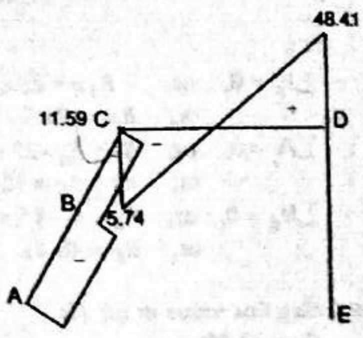
SF at $AB = 21.7 \sin 60.25^0 + 23.48 \sin 29.75^0 = 30.74\ kN$

$BC = 30.74 - 25\ kN$

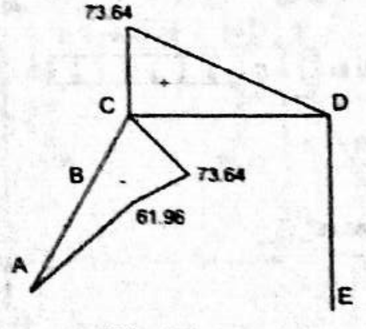
$C = -11.59\ kN$, and at $D = +48.41\ kN$



d) Axial force in (kN)



e) Shear force in (kN)



f) Bending moment (kN)

BM as drawn from the values taken from the free body diagram.

6.2 Three Hinged Arch

Beams are designed to resist maximum bending moment, which in general occurs at the centre. When the span of a beam is large, an engineer face a number of problems. First of all, the bending moment at the centre increases significantly as it is proportional to the square of span. The increase of bending moment demands bigger section, which again adds dead load to the structure causing moment to increase further. In addition, as the bending moment varies along the section of the beam, only the section at the centre of the beam gets stressed to maximum and other sections are generally under stressed. This means that the material of the beam is not fully utilized and the design becomes uneconomical.

A good solution to the above problem is to use an arch structure. An arch is nothing but a curved structure having convexity upwards and supported at ends. It may be subjected to vertical, horizontal or even inclined loads. The loads are transferred to the abutments partly by bending and axial compression Due to the curved nature of the structure, the bending moment in the arch structure is considerably reduced. The sections of an arch are subjected to normal thrust, radial shear and bending moment as shown in Fig. (6.7)

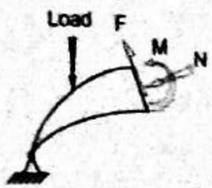


Fig. 6.7

The top most point in an arch is called crown and the vertical distance from support to this point is called rise. The support A and B of the arch point are known as springing. The central line indicated by dotted line is the axis of the arch as shown in Fig. (6.8)

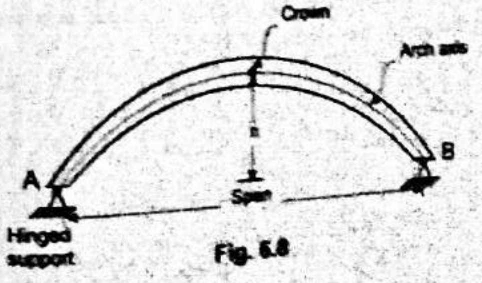


Fig. 6.8

As arch may be circular or parabolic according to its shape, it may further be divided into three categories to the support condition and the additional condition provided along the member. They are shown below.

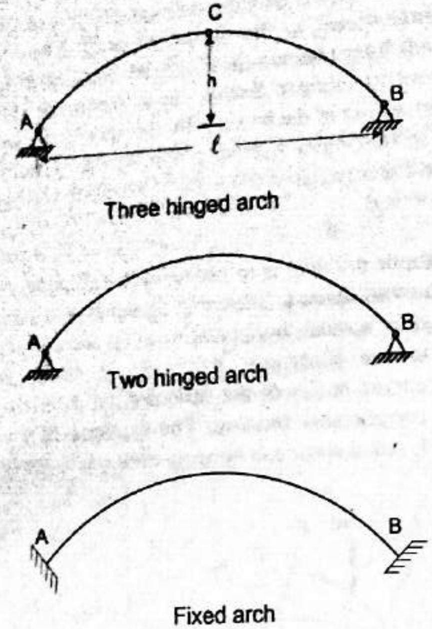


Fig. 6.9

The three-hinged arch shown above is a determinate structure whereas two hinged and fixed arches are indeterminate structures.

6.3 Line of Thrust

If the magnitude, direction and point of action of the thrust T are known at all the section of an arch, then a line of thrust, which is a funicular polygon, can be drawn. The bending moment at any section of the arch is then given by the product of horizontal component of the thrust and the ordinate between the arch axis and the line of thrust. i.e. $BM = T\cos\theta \times AB$

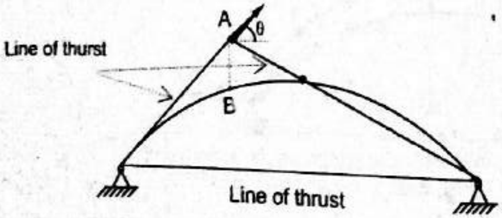


Fig. 6.10

As bending moment is proportional to ordinate AB (Eddy's theorem), the shape of the arch axis is chosen such that it coincides with the line of thrust. In that case, the bending moment in the arch will be zero and one may just design an arch for axial force. When uniformly distributed load is applied along the span of an arch, the bending moment diagram and the line of thrust will be parabolic. If now the arch can be constructed in parabolic shape, the line of thrust coincides with the axis of the arch and bending moment will be zero.

It is also to be noted that when the line of thrust is above the arch axis, it causes sagging moment, otherwise it causes hogging moment at that point.

6.4 Three Hinged Circular Arch

Let ACB in the Fig. represents a three hinged circular arch of radius R. D is a point in the arch with the co-ordinate (x, y). Completing the circle from the segment (arch) and from the property of circle, the radius R of the circular arch of span ℓ and rise h may be found as

$$\frac{\ell}{2}\times\frac{\ell}{2} = h(2R-h)$$

or, $$R = \frac{\ell^2}{8h} + \frac{h}{2}$$

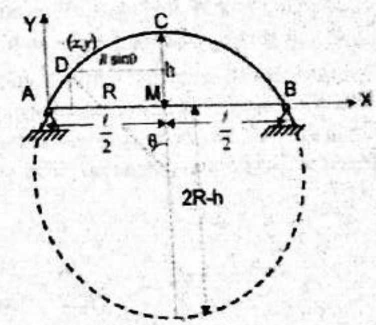


Fig. 6.11

Taking A as origin, the coordinates of any point D on the arch in Fig. (6.11) may be defined as

$$x = \left[\frac{\ell}{2} - R\sin\theta\right]$$

and $$y = R\cos\theta - (R-h)$$
$$= h - R(1-\cos\theta)$$

6.5 Three Hinged Parabolic Arch

The equation of a parabola, with origin at the left hand hinge is given by $y = kx(\ell - x)$

Putting, $y = h$ at $x = \frac{\ell}{2}$, we get

$$h = k\frac{\ell}{2}\left(\ell - \frac{\ell}{2}\right)$$

or, $$k = \frac{4h}{\ell^2}$$

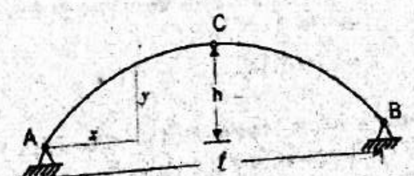


Fig. 6.12

$\therefore y = \frac{4h}{\ell^2}x(\ell - x)$ Equation of parabolic arch.

6.6 ANALYSIS OF THREE-HINGE ARCH

The following are the steps used for the analysis of arches.

1) Draw an arch and the reactions at the supports. It is usually better to choose a correct direction of the reactions as shown below.

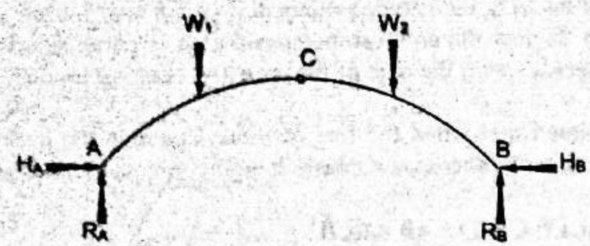


Fig. 6.13

2) Apply three equilibrium equations
$\sum F_x = 0,\ R_y = 0,\ M_A$ or $M_B = 0$
to determine first the vertical reactions. Apply additional equation $\sum M_c = 0$ for the either half of the arch to find the horizontal reaction.
3) Use arch equation to find vertical ordinate y and also the arch angle θ. Now the moment can be computed once vertical ordinate is obtained.
4) Draw a correct free body diagram for the required section and resolve the forces to find normal thrust and radial shear at a section as shown in Fig. (6.14).

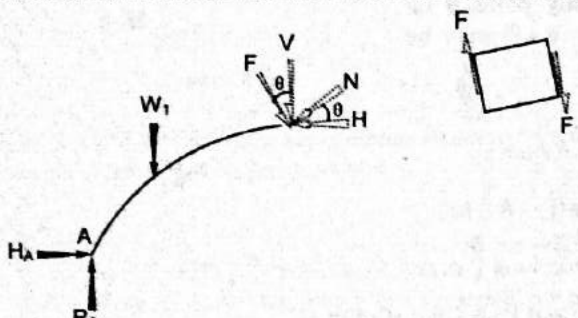


Fig. 6.14

Note that all the forces shown are in their positive sense in the cases

$$N = V\sin\theta + H\cos\theta$$

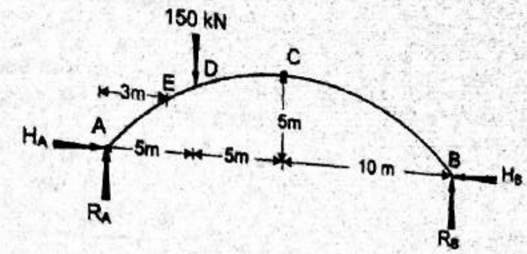
$$F = V\cos\theta - H\sin\theta$$

Example # 6.6 A circular arch of span 20 m with a central rise of 5m is hinged at the crown and springing. It carries a point load of 150 *kN* at 5 *m* from the left support calculate.

(i) The reaction at the supports
(ii) The reaction at crown

(iii) Moment at 3 *m* from the left support.

Sol^n^. The arch with its support reactions is given below



Since no horizontal forces are applied, the reactions at *A* and *B* are equal and let it be *H*.

Apply $\sum Fy = 0$, or, $R_A + R_B - 150 = 0$

or, $R_A + R_B = 150$ .... (i)

$\sum M_A = 0$, $R_B \times 20 - 150 \times 5 = 0$

or, $R_B = \dfrac{750}{20} = 37.5kN$. Ans.

Applying the value of R_B in Eq. (i), we get,

$$R_A = 150 - 37.5 = 112.5\ kN \text{ Ans.}$$

$\sum M_c = 0$, Considering right half of the arch only we get,

$$R_B \times 10 - H \times 5 = 0$$

or, $H = \dfrac{37.5 \times 10}{5} = 75\ kN$ Ans.

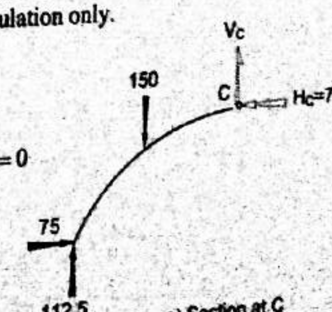
Considering the equilibrium of the left half of the arch, we can get the reaction at crown *C* by mental calculation only.

$\sum F_x = 0,\ 75 - H_c = 0$

$H_c = 75$

$\sum F_y = 0,\ 112.5 - 150 + V_c = 0$

or, $V_c = 37.5$



a) Section at C

To find moment at 3 *m* from the left end, we first need to find the vertical ordinate *y* for the section, which is given by the equations of circular arch.

$$R=\frac{\ell^2}{8h}+\frac{h}{2}$$
$$=\frac{20^2}{8\times5}+\frac{5}{2}=12.5$$

and $x=\frac{\ell}{2}-R\sin\theta$

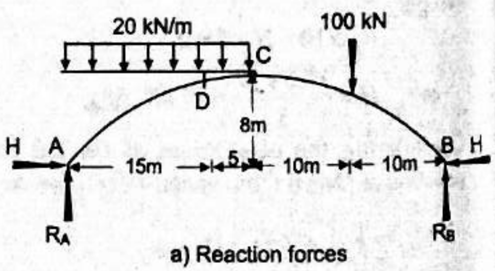
or, $3=\frac{20}{2}-12.5\sin\theta$

or, $\theta=34.06^0$

Now using $y=R\cos\theta-(R-h)$, we get

$$=12.5\cos 34.06^0-(12.5-5)$$
$$=2.85$$
$$M_{3m}=H\times y-R_A\times x$$
$$=75\times2.85-112.5\times3$$
$$=-124\ kN-m\ \underline{\text{Ans.}}$$

Example # 6.7 A three hinged circular arch hinged at springing and crown points has a span of 40 *m* and central rise of 8 *m*. It carries a uniformly distributed load 20 *kN/m* over the left half of the span together with a concentrated load of 100 *kN* at 30 *m* from the left end. Find the reaction at the support, bending moment, normal thrust and shear at a section 15 *m* from the left support.



a) Reaction forces

Sol^n. The arch with support reactions is shown in figure. Since no horizontal forces are applied

$H_A=H_B=H$

$\Sigma F_y=0,$

or, $R_A+R_B-20\times20-100=0$

or, $R_A+R_B=500$ .... (i)

$\Sigma M_A=0,$ or, $R_B\times40-100\times30-20\times\frac{20^2}{2}=0$

or, $R_B=175\ kN$. <u>Ans.</u>

Substituting this value in Eq. (i) we get,

$R_A=500-175=325\ kN$. <u>Ans.</u>

$\Sigma M_c=0$, Considering right half portion of the arch only,

$$R_B\times20-H\times8-100\times10=0$$

or, $H=\frac{175\times20-100\times10}{8}=312.5\ kN$. <u>Ans.</u>

To find moment at 15 *m* from the left end, we first need to find the vertical ordinate *y* for the section using the following equations of circular arch.

$$R=\frac{\ell^2}{8h}+\frac{h}{2}$$
$$=\frac{40^2}{8\times8}+\frac{8}{2}=29m$$

and $x=\frac{\ell}{2}-R\sin\theta$

or, $15=\frac{40}{2}-29\times\sin\theta,\quad \theta=9.93^0$

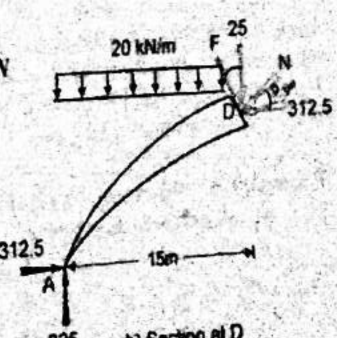
$$y=R\cos\theta-(R-h)$$
$$=29\times\cos9.93^0-(29-8)$$
$$=7.57\ m$$

Moment at $x=15m$ is given by

$$M_{10}=Hy+20\times\frac{15^2}{2}-R_A\times15$$
$$=312.5\times7.57+20\times\frac{15^2}{2}-325\times15$$
$$=-259.38\ kN\text{-}m\ \underline{\text{Ans.}}$$

Draw free body diagram of the section of the arch in which

Vertical *SF* $V=325-20\times15=25\ kN$



b) Section at D

Now,

$N=V\sin\theta+H\cos\theta$

$=25\sin9.93^0+321.5\times\cos9.93^0$

$=312.13\ kN$ <u>Ans.</u>

$F=V\cos\theta-H\sin\theta$

$=25\cos9.93^0-312.5\sin9.93^0$

$=-29.26\ kN(\uparrow)$ <u>Ans</u>

Example # 6.8 A three hinged parabolic arch of span ℓ and central rise h carries uniformly distributed load w per unit horizontal run along its whole span. Determine the reaction at the supports and bending moment at any section.

Sol^n. The arch with its support reactions is given below.

$\sum F_x = 0,$

$H_A = H_B = H$

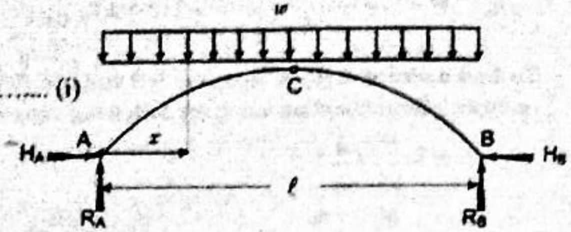
$\sum F_y = 0,$

$R_A + R_B = w\ell$ .... (i)

$\sum M_A = 0$

or, $R_B \times \ell - \frac{w\ell^2}{2} = 0$

or, $R_B = \frac{w\ell}{2}$



Substituting the value of R_B in eq. (i), we get, $R_A = \frac{w\ell}{2}$

$\sum M_c = 0$ (Considering *CB* portion only)

$$R_B\frac{\ell}{2} - H.h - w\left(\frac{\ell}{2}\right)^2.\frac{1}{2} = 0$$

or, $\frac{w\ell}{2}.\frac{\ell}{2} - \frac{w\ell^2}{8} = H.h$

or, $H.h = \frac{w\ell^2}{4} - \frac{w\ell^2}{8}$, or, $H = \frac{w\ell^2}{8h}$

The bending moment at any point *X* at a distance *x* from *A* is

$$M_x = -R_A.x + H.y + \frac{wx^2}{2}$$

or, $M_x = -\frac{w\ell}{2}x + \frac{w\ell^2}{8h} \times \frac{4hx(\ell - x)}{\ell^2} + \frac{wx^2}{2} = 0$

Thus the bending moment at any point of a three hinged parabolic arch subjected to uniformly distributed load is zero. Therefore, the external load by the arch is resisted by normal thrust only. Such types of arches are known as theoretical or linear arch.

Example # 6.9 Determine the bending moment shear and thrust at point *D* of three-hinged symmetrical parabolic arch. [T.U. 2045]

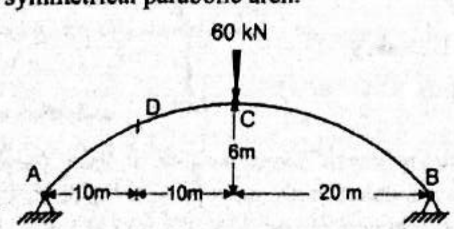


Fig. 6.15

Sol^n. The arch with its support reactions is given below.

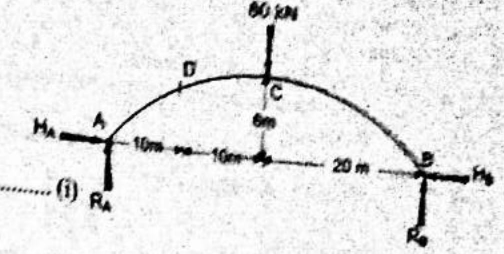
$\sum F_x = 0,$

$H_A = H_B = H$

$\sum Fx = 0,$

$R_A + R_B - 60 = 0$

or, $R_A + R_B = 60$ .... (i)



$\sum M_A = 0,$

$R_B \times 40 - 60 \times 20 = 0$

or, $R_B = 30\ kN$

Substituting this value in Eq. (i), we get

$R_B = 30\ kN$ [$\because$ due to symmetry $R_A = R_B = 30\ kN$]

$\sum M_c = 0$ and considering *CB* portion only,

$R_B \times 20 - H \times 6 = 0$

or, $H = 100\ kN$

Now we have from arch equation,

$$y = \frac{4hx}{\ell^2}(\ell - x) = \frac{4 \times 6 \times x}{40^2}(40 - x)$$

or, $y = 0.6x - 0.015x^2$ .... (ii)

$\frac{dy}{dx} = 0.6 - 0.015 \times 2x$

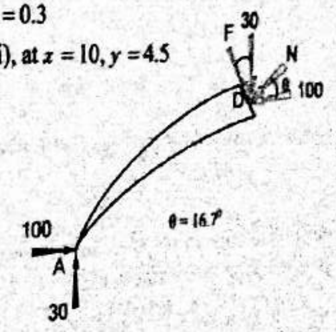
or, $\tan\theta = 0.6 - 0.015 \times 2 \times 10 = 0.3$

$\therefore$ $\theta = 16.7^0$, and from Eq. (ii), at $x = 10, y = 4.5$

$M_D = H \times 4.5 - R_A \times 10$

$= 100 \times 4.5 - 30 \times 10$

$= 150\ kN\text{-}m$ Ans.



For normal thrust and the radial shear, let us draw the free body diagram for *AD* portion as shown in the figure, Then,

$N = 100\cos 16.7^0 + 30\sin 16.7^0 = 104.40$ (←) kN Ans.

$F = 30\cos 16.7^0 - 100\sin 16.7^0 = 0$ Ans.

Example # 6.10 A three hinge parabolic arch has a span of 20 m and central rise of 5 m. It carries a concentrated load of 80 kN at a distance of 5 m from the left support.

(i) Determine support reactions

(ii) Find the maximum bending moment and plot the bending moment diagram.

(iii) Find the value of thrust and radial shear at the section 8 m. from the left hand support.

Sol^n. The arch with its supports reactions is shown in Fig (6.16).

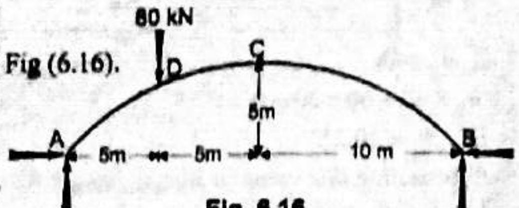


Fig. 6.16

a) $\sum F_x = 0,\ H_A = H_B = H$

$$\sum F_y = 0,$$

or, $R_A + R_B - 80 = 0$

or, $R_A + R_B = 80$ .... (i)

Take moment about A,

$$R_B \times 20 - 80 \times 5 = 0$$

or, $R_B = \dfrac{400}{20} = 20kN$ Ans.

Substituting the value of R_B in Eq. (i) we get,

$$R_A = 80 - 20 = 60\ kN.\ \text{Ans.}$$

Now, take moment about C and consider the right half of the arch only (CB), then

$$\sum M_c = 0$$

or, $R_B \times 10 - H \times 5 = 0$

or, $H = \dfrac{20 \times 10}{5} = 40kN.$ Ans.

b) The equation of arch with its origin at A is

$$y = \frac{4hx(\ell - x)}{\ell^2} \quad \text{.................................. (ii)}$$

and the maximum positive bending moment occurs under the 80 kN load i.e. at $x = 5$. Putting this in Eq. (ii)

$$y = \frac{4 \times 5 \times 5(20-5)}{20^2}$$

$$= 3.75\ m$$

Now, $M_{max} = H.y - R_A.x$

$$= 40 \times 3.75 - 60 \times 5$$

$$= -150\ kN - m.\ \text{Ans.}$$

The maximum bending moment will occur in span CB. Let it occur at a distance x from A, Then,

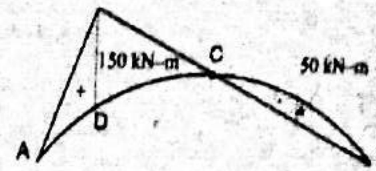
$$y = \frac{2 \times 5x(20 \times x)}{20^2} = \frac{x(20-x)}{20}$$

$$M_x = 20(20-x) - 40.\frac{x(20-x)}{20} = 400 - 60x + 2x^2$$

For maximum of M_x, $\dfrac{dM_x}{dx} = -60 + 4x = 0 \qquad \therefore x = 15\ m$

$\therefore \quad M_x = 400 - 60 \times 15 + 2 \times 15^2 = -50kN - m$ Ans.

Now the values of maximum positive and negative bending moments are known. These points are joined by straight lines (point load acting) in such a way that the point of contra-flexure (change of moment) occurs at C. The plot of bending moment is thus shown in figure.



c) For the thrust and radial shear at 8m from the left hand support, let us first draw free body diagram of the Section, $V = 60 - 80 = -20\ kN$

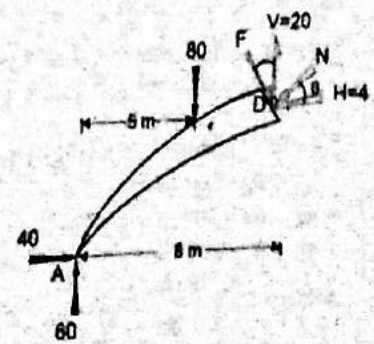
The direction of F and N are shown in their positive sense, we have,

$$y = \frac{4hx(\ell - x)}{\ell^2}$$

$$y = \frac{4 \times 5 \times x(20-x)}{20^2} = \frac{x}{20}(20-x)$$

or, $\dfrac{dy}{dx} = 1 - \dfrac{2x}{20}$

or, $\tan\theta = 1 - \dfrac{2 \times 8}{20}, \quad \therefore \theta = 11.31^0$



Normal thrust $\quad N = 40\cos 11.31^0 - 20\sin 11.31^0 = 35.3\ kN(\leftarrow)$ Ans.

$$F = -20\cos 11.31^0 - 40\sin 11.31^0 = 27.45\ kN(\uparrow)\ \text{Ans.}$$

Example # 6.11 Find the moment, shear and thrust at point D for the three hinged parabolic arch loaded as shown. [T.U. 2046]

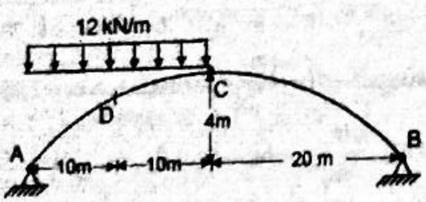
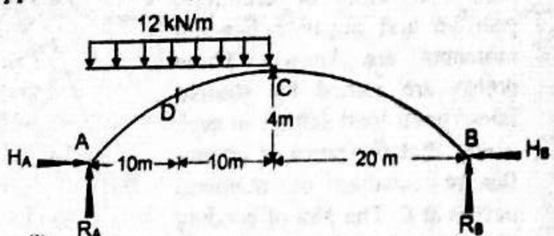


Fig. 6.16

Sol^n. The arch with its support reactions is given below



$\Sigma F_x = 0,$
$H_A = H_B = H$

$\Sigma F_y = 0,$
$H_A + H_B - 12 \times 20 = 0$

or, $R_A + R_B = 240$ .... (i)

$\Sigma M_A = 0,$

$R_B \times 40 - 12 \times \dfrac{20^2}{2} = 0, \qquad \therefore R_B = 60\ kN$

Substituting this value of R_B in Eq, (i), we get
$R_A = 240 - 60 = 180\ kN.$

$\Sigma M_c = 0$ considering CB portion only.
$R_B \times 20 - H \times 4 = 0$
or, $60 \times 20 - H \times 4 = 0$
or, $H = 300\ kN$

we have, $y = \dfrac{4 \times 4 \times x}{40^2}(40 - x)$

or, $y = 0.4x - 0.01x^2$ .... (ii)

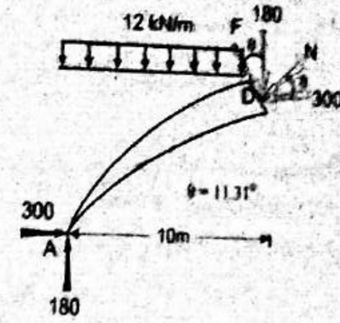
$\dfrac{dy}{dx} = \tan\theta = 0.4 - 0.01 \times 2x$

at $x = 10$, $\tan\theta = 0.4 - 0.2 = 0.2$
$\therefore \theta = 11.31^0$

Again, putting $x = 10$ in Eq. (ii)
$y = 0.4 \times 10 - 0.01 \times 10^2$
$\therefore y = 4 - 1 = 3$

Now, $M_D = H \times 3 - R_A \times 10 + 12 \times 10 \times \dfrac{10}{2}$
$= 300 \times 3 - 180 \times 10 + 600 = -300\ kN\text{-}m$ **Ans.**

For normal thrust and radial shear force at D, let us draw free body diagram of AD portion as in the figure.



$N = 300 \cos 11.31^0 + 180 \sin 11.31^0$
$= 329.47\ (\leftarrow)\ kN$ **Ans.**

$F = 180 \cos 11.31^0 - 300 \sin 11.31^0$
$= 117.67^0\ (\downarrow)\ kN$ **Ans.**

Example # 6.12 A parabolic arch hinged at springings and crown has a span of 20 m. The central rise of the arch is 4 m. It is loaded with a uniformly distributed load of 2 kN/m on the left 8 m length. Calculate

(a) The direction and magnitude of reaction at the hinge.
(b) The BM, normal thrust and shear at 4 m from the left end
(c) Maximum positive and negative bending moments.

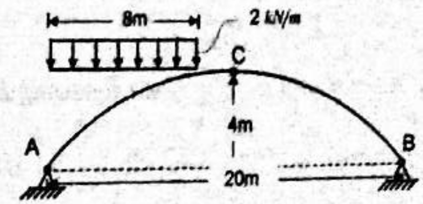
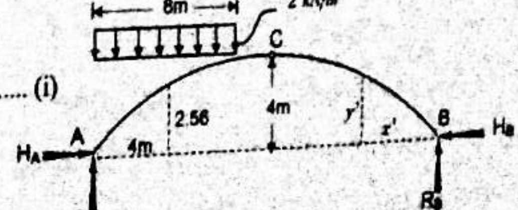


Fig. 6.17

Sol^n. Let us first draw the reactions of the arch



$\Sigma F_x = 0,$
or, $H_A - H_B = 0$ .... (i)

$\Sigma F_y = 0,$
or, $R_A + R_B - 8 \times 2 = 0$
or, $R_A + R_B = 16$ .... (ii)

$\Sigma M_A = 0$

or, $R_B \times 20 - 2 \times \dfrac{8^2}{2} = 0, \qquad \therefore R_B = 3.2\ kN$

Substituting this value in Eq. (ii),
We get, $R_A = 12.8\ kN$

Again, apply $\sum M_c = 0$ and consider CB portion of the arch only, we get,

$R_B \times 10 - H_B \times 4 = 0$

or, $H_B = \dfrac{3.2 \times 10}{4} = 8kN.$

Substituting this value in Eq. (ii) we get, $H_A = 8\ kN$

The reaction at hinge A is thus,

Resultant reaction at $A = \sqrt{R_A^2 + H_B^2}$

at $A = \sqrt{12.8^2 + 8^2} = 15.09kN.$ Ans.

The direction of the resultant is given by

$$\tan\theta_A = \frac{R_A}{H} = 1.6$$

$$\therefore \theta_A = 58^0 \quad \text{Ans.}$$

Similarly,

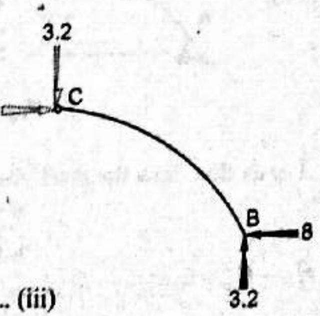
Resultant at point $B = \sqrt{3.2^2 + 8^2} = 8.62kN.$

Direction $\theta_B = \tan^{-1}\dfrac{R_B}{H} = \tan^{-1}\dfrac{3.2}{8} = 21.8^0$ Ans.

Reaction at C is same as of reaction at B as seen by the following free body diagram.

So, resultant at $C = 8.62\ kN.$ Ans.

Direction, $\theta_c = 21.8^0$ Ans.



(b) Equation of parabola is $y = \dfrac{4hx}{\ell^2}(\ell - x)$

at $h = 4,\ \ell = 20$

$$y = \frac{4\times 4}{400}x(20 - x)$$

or, $y = 0.8x - 0.04x^2$ .... (iii)

or, $\dfrac{dy}{dx} = 0.8 - 0.04 \times 2x$ .... (iv)

Now, from Eq. (iii),

at $x = 4m,\ y = 2.56m$

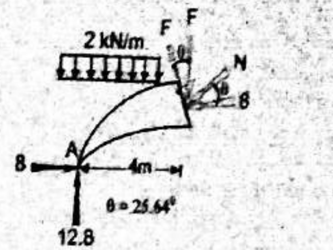
and from (iv)

$$\tan\theta = \frac{dy}{dx} = 0.48, \qquad \therefore \theta = 25.64^0$$

Now, moment at 4 m from A is

$$M_A = H_A \times 2.56 + 2\times\frac{4^2}{2} - R_A \times 4$$
$$= 8\times 2.56 + 16 - 12.8\times 4$$
$$= 14.72\ kNm \quad \text{Ans.}$$

Normal thrust and radial shear are calculated by the following free body diagram of the section AD.



$= (12.8 - 2\times 4) = -4.8$

$= 8\cos 25.64^0 + 4.8\sin 25.64^0$

$= 2.29(\leftarrow)\ kN. = 9.29$ Ans.

$= 4.8\cos 25.64^0 - 8\sin 25.64^0$

$= 0.87(\downarrow)kN.$ Ans

) Maximum positive and negative bending moments

Maximum +ve BM occurs somewhere at section AC

Let it is x distance away from A

Then, $M_x = -12.8x + 2\dfrac{x^2}{2} + 8y$ .... (v)

For M_x to be maximum, $\dfrac{dMx}{dx} = 0$

$$-12.8 + 2x + 8.\frac{dy}{dx} = 0 \text{ .... (vi)}$$

Substituting value of $\dfrac{dy}{dx}$ from Eq (iv), we get

$$-12.8 + 2x + 8(0.8 - 0.08x) = 0$$

or, $x = 4.71$

For +ve M_{max}

From Eq. (v), $M_x = -12.8x + x^2 + 8y$

Put $y = (0.8x - 0.04\times x^2)$ Eq. (iii)

and $x = 4.71$

We get, $M_{max} = -12.8\times 4.71 + 4.71^2 + 8\times(0.8\times 4.71 - 0.04\times 4.71^2)$

$= 15.06\ kN - m$ (Anti clockwise) Ans.

Similarly, maximum – ve bending moment occurs at the, section CB.
Let, x' is the distance from B left to the arch section at E, where BM is maximum, $M'_x = R_B x' - H_B y'$

Arch equations will be similar to Eq. (iii) and (iv)
Which are, $y' = 0.8x' - 0.04x'^2$

$$\frac{dy'}{dx'} = 0.8 - 0.04 \times 2x'$$

M_x to be max, $\frac{dM_x}{dx} = 0$

or, $R_B - H_B \cdot \frac{dy'}{dx'} = 0$

Substituting the values

$$3.2 - 8 \times (0.8 - 0.08x') = 0$$

or, $3.2 - 6.4 + 0.64x' = 0$

or, $x' = 5m$

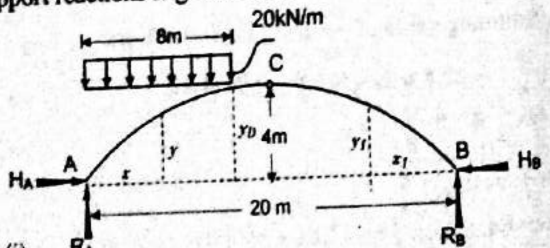
max – ve moment $= R_B x' - H_B y'$

$$= 3.2 \times 5 - 8 \times (0.8 \times 5 - 0.04 \times 5^2)$$

$8\ kN - m$ ( ) Ans.

Example # 6.13 A parabolic three-hinged arch has a span of 20 m and rise 4 m. It is loaded with a uniformly distributed load of 20 kN/m for a length of 8 m from the left end support. Draw the bending moment diagram and find the position and magnitude of maximum BM over the arch.

Sol: The arch with its support reactions is given below



$\sum F_x = 0,$

$H_A = H_B = H$

$\sum Fy = 0,$

$R_A + R_B - 20 \times 8 = 0$

or, $R_A + R_B = 160$ .... (i)

$\sum H_A = 0,$ or, $R_B \times 20 - 20 \times \frac{8^2}{2} = 0$

or, $R_B = \frac{20 \times 8^2}{2 \times 20} = 32kN$

Substituting the value of R_B in Eq. (i), we get

$$R_B = 160 - 32 = 128\ kN$$

$\sum M_c = 0$ considering CB portion of the arch only,

$$R_B \times 10 - H \times 4 = 0$$

or, $H = \frac{32 \times 10}{4} = 80kN$

$$M_D = -R_A \times 8 + H \times y_D + 20 \times \frac{8^2}{2} \qquad \left[y_D = \frac{4hx}{\ell^2}(\ell - x) = \frac{4 \times 4 \times 8}{20^2}(20 - 8) = 3.84m \right]$$

$$= -128 \times 8 + 80 \times 3.84 + 20 \times \frac{8^2}{2}$$

$= -76.8$ (-ve sign shows clockwise moment)

BM diagram is as follows
Let maximum positive BM occur at x distance from A.
As seen from the moment diagram,

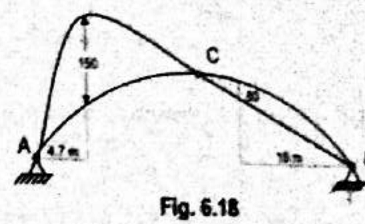


Fig. 6.18

We have, $y = \frac{4hx}{\ell^2}(\ell - x)$

or, $y = \frac{4 \times 4x}{20^2}(20 - x)$

or, $y = 0.8x - 0.04x^2$

$$M_x = H.y + \frac{20x^2}{2} - R_A x$$

$$= 80(0.8x - 0.04x^2) + 10x^2 - 128x$$

$$= 6.8x^2 - 64x$$

For M_x to be maximum,

$\frac{dM_x}{dx} = 0,$ or, $6.8 \times 2x - 64 = 0$

or, $x = \frac{64}{6.8 \times 2} = 4.71\ m$

$\therefore M_{max}(+ve) = 6.8 \times 4.71^2 - 64 \times 4.71$

$= 150.59kN - m.$ Ans.

Let negative bending moment occur at a distance x from B, such that

$$M_{x_1} = R_B \times (\ell - x_1) - Hy_1$$

$$= 32(20 - x_1) - 80(0.8x_1 - 0.04x_1^2)$$

$$= 640 - 96x_1 + 3.2x^2$$

$$\left(y_1 = \frac{4hx_1}{\ell^2}(\ell - x_1) = \frac{4 \times 4x_1}{20^2}(20 - x_1),\quad y_1 = 0.8x_1 - 0.04x_1^2 \right.$$

M_{x_1} is maximum if $\frac{dM_{x_1}}{dx_1} = 0,$

or, $-96+3.2\times 2x_1=0$

or, $x_1=\frac{96}{6.4}=15\ m$

$M_{x_1}=640-96\times 15+3.2\times 15^2=-80\ kN-m$ Ans.

BMD is shown in Fig. (6.18)

Example # 6.14 Find the stresses at a section 5 *m* from the left hand support of a three-hinged parabolic arch with supports at the same level

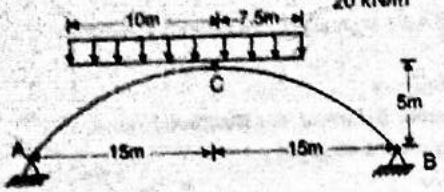


Fig. 6.19

Sol^n. The arch with its support reactions is given below

$\sum Fx=0,\quad H_A=H_B=H$

$\sum Fy=0,\quad R_A+R_B-20\times 17.5=0$

or, $R_A+R_B=350$ .... (i)

$\sum M_A=0,\quad R_B\times 30-20\times 17.5\left(5+\frac{17.5}{2}\right)=0$

or, $R_B=160.42\ kN$

Substituting the value of R_B in Eq. (i), we get

$R_A=350-160.42=18958\ kN$

$\sum M_c=0$, Considering *CB* portion only,

$R_B\times 15-H\times 5-20\times\frac{7.5^2}{2}=0$

or, $H=\frac{160.42\times 15-20\times\frac{7.5^2}{2}}{5}=368.76\ kN$

$\left[\begin{array}{l} y=\frac{4\times 5\times 5}{30^2}(30-5) \\ =2.78m \end{array}\right.$

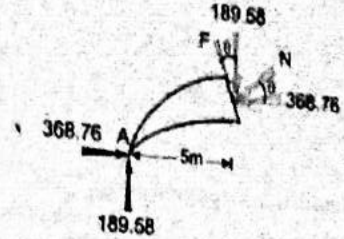
Now, moment at $x=5m$

$M_x=H.y-R_A.x$

$=368.76\times 2.78-189.58\times 5$

$=77.25kN-m$

For normal thrust and radial shear let us first draw the free body diagram at 5 *m* section.



We have,

$y=\frac{4\times 5x}{30^2}(30-x)=0.667x-0.022x^2$

$\frac{dy}{dx}=0667-0.022\times 2x=\tan\theta$

or, $\tan\theta=0.667-0.022\times 2\times 5=0.447$

$\theta=24.08^0$

Now,

$N=H\cos\theta+V\sin\theta$

$=368.76\cos 24.08^0-189.58\sin 24.08^0$

$=414.02\ kN\ (\leftarrow)$ Ans.

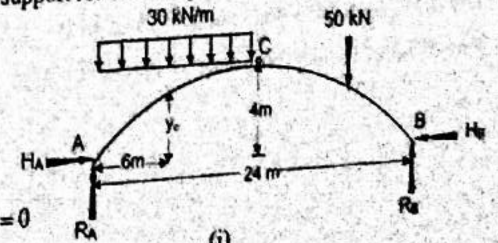
$F=V\cos\theta-H\sin\theta$

$=189.58\cos 24.08^0-368.76\sin 24.08^0$

$=22.62\ (\downarrow)$ Ans.

Example # 6.15 *A three-hinged parabolic arch hinged at the supports and at the crown has a span of 24 m and a central rise of 4 m. It carries concentrated load of 50 kN at 18 m from left support and uniformly distributed load of 30 kN/m over the left half portion. Determine the moment, thrust and radial shear at a section 6m from the left support.*

Sol^n. The arch with its support reactions is given below



$\sum F_x=0,$

$H_A=H_B=H$

$\sum F_y=0,$

$R_A+R_B-30\times 12-50=0$

or, $R_A+R_B=410$ .... (i)

$$\sum M_A = 0, \quad \text{or, } R_B \times 24 - 50 \times 18 - 30 \times \frac{12^2}{2} = 0$$

or, $R_B = 127.5\ kN$.

Substituting the value of R_B in Eq. (i), we get

$$R_A = 40 - 127.5 = 282.5\ kN$$

$\sum M_c = 0$, Considering CB portion only,

$$R_B \times 12 - H \times 4 - 50 \times 6 = 0$$

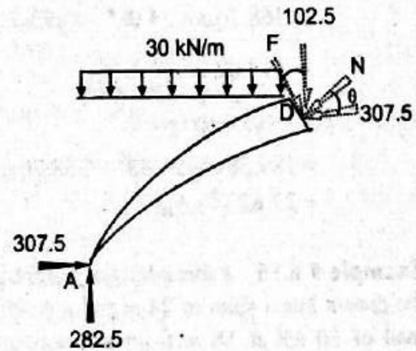
or, $H = \dfrac{127.5 \times 12 - 50 \times 6}{4} = 307.5\ kN$

At 6 m from the support

$$M_6 = -R_A \times 6 + H.y_D + 30 \times \frac{6^2}{2} \qquad \text{where } y = \frac{4hx(\ell - x)}{\ell^2}$$

$$M_6 = -282.5 \times 6 + 307.5 \times 3 + 30 \times \frac{6^2}{2} \qquad \text{at } x = 6, y_D = 3m$$

$$= -232.5\ kN - m.$$



Vertical shear

$V = R_A - 30 \times 6$

$= -282.5 - 30 \times 6 = -462.5\ kN$

Angle θ is give by

$$y = \frac{4hx(\ell - x)}{\ell^2}$$

$$\frac{dy}{dx} = \tan\theta = \frac{4h(\ell - 2x)}{\ell^2}$$

or, $\tan\theta = \dfrac{4 \times 4(24 - 2 \times 6)}{24^2}$

$\therefore \quad \theta = 18.44^0$

$N = 307.5\cos 18.44^0 + 462.5\sin 18.44^0 = 324.13(\leftarrow)$ Ans.

$F = -307.5\sin 18.44^0 + 462.5\cos 18.44^0 = 0(\uparrow)$ Ans.

Example # 6.16 Find the values of bending moment shear force and thrust at a point D for the parabolic arch shown in the figure.

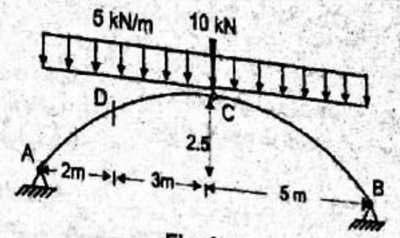
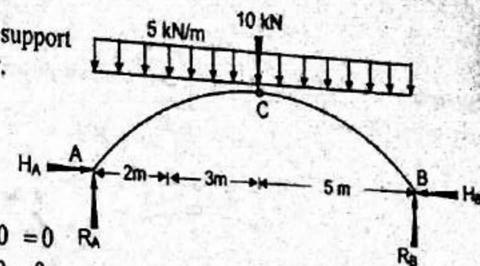


Fig. 6.20

Sol^n. The arch with its support reactions is given below.



$\sum F_x = 0,$

$H_A = H_B = H$

$\sum F_y = 0,$

$H_A + H_B - 10 - 5 \times 10 = 0$

or, $R_A + R_B - 10 - 5 \times 10 = 0$ .... (i)

$$\sum M_A = 0, \quad R_B \times 10 - 10 \times 5 - 5 \times \frac{10^2}{2} = 0$$

Substituting the value of R_B in Eq. (i), we get $R_A = 30\ kN$

$\sum M_c = 0$ considering CB portion only,

$$R_B \times 5 - H \times 2.5 - 5 \times \frac{5^2}{2} = 0$$

or, $30 \times 5 - 2.5H - \dfrac{125}{2} = 0$ or $H = 35\ kN$

We have the arch equation

$$y = \frac{4 \times 2.5x}{10^2}(10 - x)$$

or, $y = x - 0.1x^2$ .... (ii)

or, $\dfrac{dy}{dx} = 1 - 0.1 \times 2x$

at $x = 2$, $\dfrac{dy}{dx} = \tan\theta = 1\text{-}0.1 \times 2 \times 2 = 0.6$

$\therefore \quad \theta = 30.96^0$

Again, putting $x = 2$ in Eq. (ii)

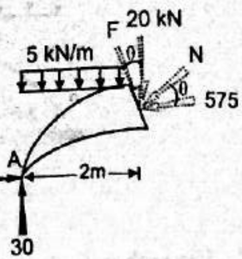
$$y = 2 - 0.1 \times 2^2 = 1.6\ m$$

Now, $M_D = H \times 1.6 - R_A \times 2 + 5 \times \frac{2^2}{2}$

$= 35 \times 1.6 - 30 \times 2 + 5 \times 2$

$= 6$ *kN-m.*

For normal thrust and radial shear force at D,
The free body diagram of AD is shown in figure.



$N = 35 \cos 30.92^0 + 20 \sin 30.92^0$

$= 40.3$ (←) Ans.

$F = 20 \cos 30.92^0 - 35 \sin 30.92^0$

$= -0.827$ (↑) Ans.

Example # 6.17 Calculate the stress resultant at point D for the parabolic arch shown below. [T.U. 2051]

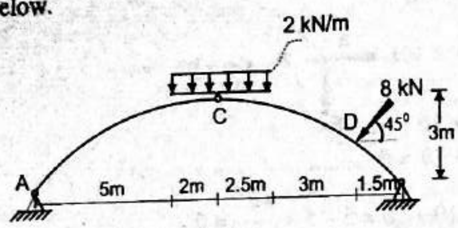
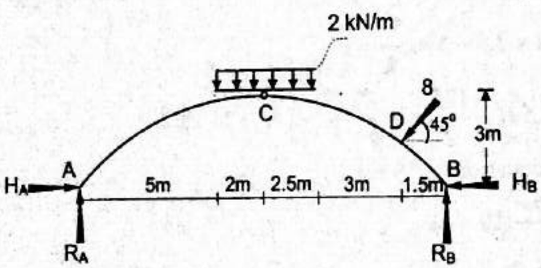


Fig. 6.21

Sol^n. The arch with its support reactions is given below.



$\sum F_x = 0, \qquad H_A - H_B + \cos 45^0 = 0$ or $H_A - H_B = -5.66$ .... (i)

$\sum F_y = 0, \qquad R_A + R_B - 2 \times 4.5 - 8 \sin 45^0 = 0$

or, $R_A + R_B = 14.66$ .... (ii)

Since the force of 8 *kN* is acting at D, we first need to find the vertical ordinate of this point before proceeding to $\sum M_A = 0$,
We have,

$$y = \frac{4hx}{\ell^2}(\ell - x)$$

$$= \frac{4 \times 3 \times 1.5}{14^2}(14 - 1.5) = 1.15m.$$

$\sum M_A = 0, R_B \times 14 + 8 \cos 45^0 \times 1.15 - 8 \sin 45^0 \times 12.5 - 2 \times 4.5\ 5 + \frac{4.5}{2} = 0$

or, $R_B = 9.25$ *kN*

Substituting the value of R_B in Eq. (i), we get $R_A = 5.41$ *kN*
$\sum Mc = 0$, considering AC portion only,

$H_A \times 3 - R_A \times 7 + 2 \times 2\ (1) = 0$

or, $H_1 = \frac{5.41 \times 7 - 4}{3} = 11.29$ *kN*

Substituting the value of H_A in Eq. (i) we get,

$H_B = H_A + 5.66$

$= 11.29 + 5.66 = 16.95$ *kN.*

Example # 6.18 For the three-hinged frame shown as loaded, find the reaction at the hinges and the position and magnitude of the maximum bending moment. [T.U. 2046]

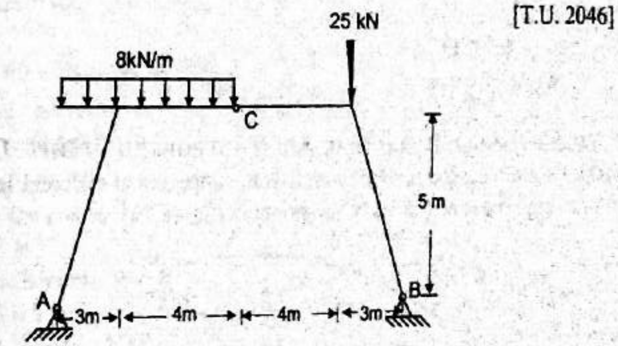


Fig. 6.22

Here,

$R_A + R_B = 8 \times 7 + 25$

or, $R_A + R_B = 81$ *kN* .... (ii)

Now, Taking moment at A,

$\sum M_A = 0, \qquad R_B \times 14 - 25 \times 11 - 8 \times 7 \times \frac{7}{2} = 0$

∴ $R_B = 33.64$ *kN.* (↑)

$\therefore$ Putting value of R_B in Eq. (ii)

$R_A = (81 - 33.64)$ kN

$= 47.36$ kN. (↑)

Again considering moment at C, Taking CB portion only.

$\sum M_c = R_B \times 7 - 25 \times 4 - H \times 5$

or, $0 = 33.647 \times 7 - 25 \times 4 - H \times 5$

or, $H = 27.1$ kN (←)

Now, Taking AC portion

$M_x = R_A \times x - H.y$

$= 47.36x - 27.1 \times y$

$M_x = 47.36x - 27.1 \times y$

at $x = 0, \quad y = 0$

$M_x = 47.36x - 27.1 \times (x \tan 59.04)$

$\therefore$ $y = x \tan 59.04^0$

$\frac{d}{dx}(M_x) = 47.36 - 27.1.\tan 59.04^0$

$= 2.186$ (+ve value)

So, in portion, AC

$M_0 = 0$

$M_{15} = 48.82$ kN-m

$M_3 = 6.58$ kN-m

6.7 THREE HINGED PARABOLIC ARCH SUPPORTED AT DIFF. LEVELS

Consider a three-hinged parabolic arch ACB, supported at different levels at A and B having hinges at A, B and C as shown in Fig. (6.23)

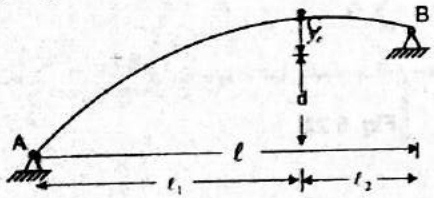


Fig. 6.23

Let ℓ = Span AB of the arch,

y_c = Height of the crown C from the upper support B

d = Difference between the levels of the two supports,

ℓ_1 = Horizontal length between A and C, and

ℓ_2 = Horizontal length between C and B

Let the arch ACB be extended to D, such that A and D are at the same level. Let the span of the imaginary arch ACD be L, such that L is equal to $2\ell_1$. A little consideration will show, that the rise of any point I on the arch axis at a distance x from the support is

$$y = \frac{4(y_c + d)}{\ell^2} x(\ell - x)$$

The reaction at the two ends A and B will have both vertical and horizontal components. When an arch is subjected to vertical loads only, the horizontal components at the two supports will be equal and opposite.

The vertical and horizontal components R_A, R_B and H may be found out by taking moments about any support; and then by taking moments about C and equating the same to zero.

Example # 6.19 A three-hinged parabolic arch of 60 m span is loaded as shown in Fig. (6.24). Find the bending moment at a point P, 20 m from the left hand support.

Sol^n.

Given,

Span $\ell = 60$ m

Horizontal distance between the crown and the left support A

$\ell_1 = 60 - 20 = 40$ m

Horizontal distance between the crown and right support B,

$\ell_2 = 20$ m

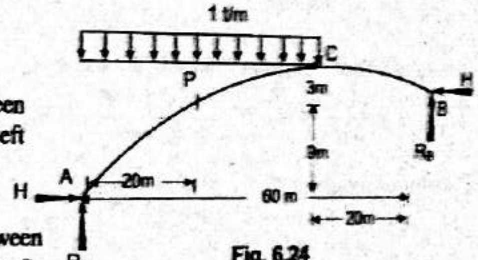


Fig. 6.24

Height of the crown C from the higher support,

$y_c = 3$ m

Difference between the levels of the two supports,

$d = 9$ m

Let, ℓ = Span of the imaginary arch ACD, such that A and D are at the same Level.

R_A = Vertical reaction at A, and

R_B = Vertical reaction at B,

We know that the rise of arch, at any section X, at a distance x from A,

$$y = \frac{4(y_c + d)}{\ell^2} x(\ell - x) = \frac{4(3+9).x}{\ell^2}(\ell - x)$$

$$= \frac{48x}{\ell^2}(\ell - x) \quad \text{......... (i)}$$

We also know that when $x = 60$, $y = 9$. Substituting these values of x and y in the above equation,

$$9 = \frac{48\times 60}{\ell^2}(\ell - 60) = \frac{2{,}880}{\ell^2}(\ell - 60)$$

$$= \frac{2{,}880}{\ell} - \frac{1{,}72{,}800}{\ell^2}$$

$$\therefore \quad 9\ell^2 - 2{,}880\ell + 1{,}72{,}800 = 0$$

or, $\ell^2 - 320\ell + 19{,}200 = 0$

Solving the quadratic equation for ℓ,

$$\ell = \frac{320 \pm \sqrt{320^2 - 4\times 19{,}200}}{2}$$

$$= \frac{320 - 160}{2} = 80\ m$$

Substituting the value of ℓ in Eq (i)

$$y = \frac{48x}{80^2}(80 - x)$$

$\therefore$ Rise of arch at P i.e. at a distance of 20 m from A,

$$y = \frac{48\times 20}{80^2}(80-20) = \frac{48\times 20\times 60}{80\times 80} = 9\ m$$

Taking moment about A,

$$R_B \times 60 + H\times 9 = 1\times 40\times 20$$

$$H = \frac{800 - 60R_B}{9} \quad \text{.. (ii)}$$

Since the bending moment at the crown C is zero, therefore

$$3H = 20R_B$$

Substituting the value of H in the above equation,

$$3\left(\frac{800 - 60R_B}{9}\right) = 20R_B$$

$$\frac{800}{3} - 20V_B = 20R_B$$

$$\therefore \quad 40R_B = \frac{800}{3}$$

or, $R_B = \frac{800}{3}\times\frac{1}{40} = \frac{20}{3}\ t$

$$\therefore \quad R_A = 40 - \frac{20}{3} = \frac{100}{3}\ t$$

Substituting the value of R_B in equation (ii),

$$H = \frac{800 - 60\times\frac{20}{3}}{3} = \frac{800-400}{9} = \frac{400}{9}\ t$$

$\therefore$ Bending moment at P,

$$M_P = R_A\times 20 - H\times 9 = \frac{100}{3}\times 20 - \frac{400}{9}\times 9 - 1\times 20\times 10$$

$= 66.64$ t-m **Ans.**

6.8 Influence Lines Diagram for Three-Hinged Arches

The figure on the right hand side shows an arch of span ℓ and rise h.

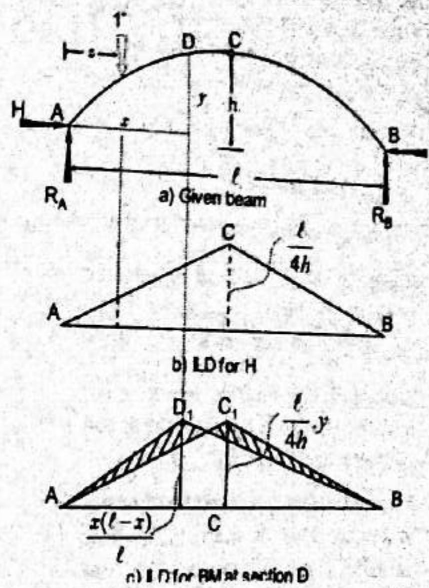


Fig. 6.25

(a) *ILD for horizontal reaction*

Take moment about point C, then

$$\Sigma M_C = 0, \quad H\times h = R_B\times\frac{\ell}{2}$$

or, $H = \frac{R_B\times\ell}{2.h}$ .... (i)

Let at any instance the unit load is at a distance Z from A. obviously at the position

$$R_B = \frac{1.z}{\ell} = \frac{z}{\ell}$$

Substituting the value of R_B in Eq. (i),

we get, $H = \frac{z}{2h}$

at $z = 0$, $H = 0$

at $z = \frac{\ell}{2}$, $H = \frac{\ell}{4h}$

So, as the unit load moves from A to C, the horizontal thrust will change from zero to $\frac{\ell}{4h}$. Obviously as the unit load moves from C to B, horizontal thrust will change $\frac{\ell}{4h}$ from to zero. The diagram is shown in Fig.(6.25-b).

(b) *ILD for BM at D*

Take moment about D, $M_D = R_B(\ell - x) - Hy$

But, $R_B(\ell - x)$ = Beam moment at D.

It means that the expression $R_B(\ell - x)$ is same as that of simply supported beam of equal span with the arch. Thus if the influence line of simply supported beam is subtracted with the influence line for horizontal thrust multiplied by y, we obtain *ILD* for moment at D for the arch. The resulting diagram is shown in Fig. (6.15-c)

c) *ILD for Normal thrust and Radial shear at the section*

Consider the free body diagram of the section AD of the arch. The unit force is in between A and D. Note that the balancing vertical reaction R_B will act upward.

Normal thrust (N) for the section is given by

$N_{AD} = H\cos\theta - R_B \sin\theta$ .... (ii)

(With convention adopted for an arch)

Similarly when the load is between D and B, the thrust is given by

$N_{DB} = H\cos\theta + R_A \sin\theta$ ...... (iii)

Now with the help of relations (ii) and (iii) the influence line diagram is drawn as follows.

First we draw the influence line for $H\cos\theta$. This as a triangle whose altitude is $(\ell/4h)\cos\theta$. On this diagram, we superimpose the influence line diagram for $R_B \sin\theta$ for part AD. For the part DB, we should add the influence line diagram for $R_A \sin\theta$ to the influence line diagram for $H\cos\theta$. The diagram is shown in Fig. (6.26-f). Similarly, for radial shear at D, considering Fig. (6.26-a) and (b) we obtain the expressions

$$F_{AD} = H\sin\theta + R_B\cos\theta$$

$$F_{DB} = H\sin\theta - R_A\cos\theta$$

ILD is shown in Fig. (6.24-g)

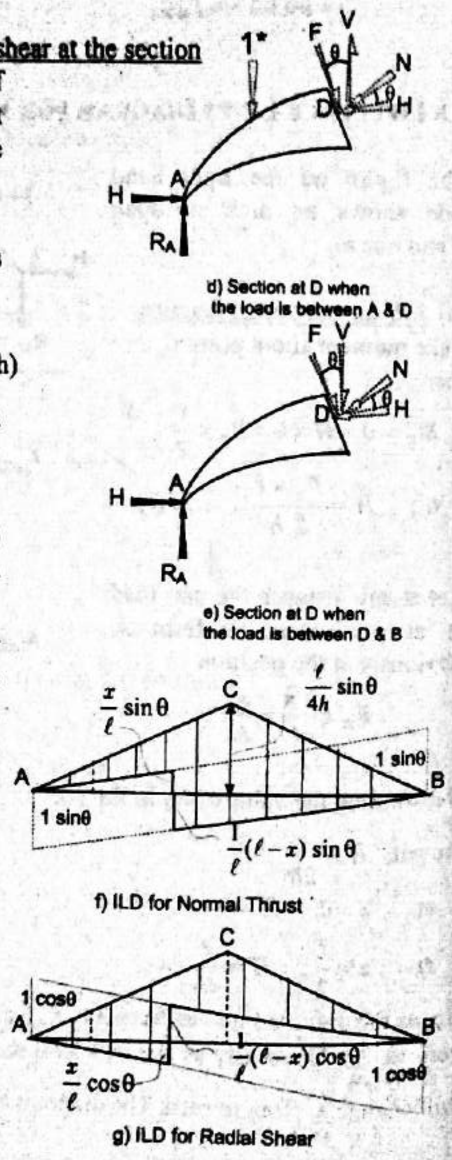


Fig. 6.26

ILD is shown in Fig. (6.24-g)

Example # 6.20 Three hinged circular arch has a span of 30 m with a central rise of 6 m. Two loads 80 kN and 200 kN roll over the arch from left to right. Draw influence line for bending moment at a section 10 m from the left hand hinge, the maximum bending moment and horizontal thrust at the section due to rolling loads.

[2058 Jeshtha]

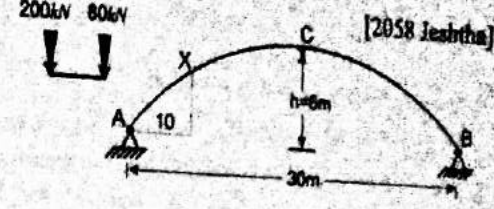


Fig. 6.27

Sol^n.

Given,

Leading load $W_1 = 80\ kN$

Trailing load $W_2 = 200\ kN$

Distance between the section and left hand support, $x = 10\ m$

Let,

h = Central rise of the arch

H = Horizontal thrust

R_A = Vertical reaction at A, and

R_B = Vertical reaction at B,

Now, from property of a circle,

$$\frac{\ell}{2} \times \frac{\ell}{2} = H(2R - h)$$

$$R = \frac{\ell^2}{8h} + \frac{h}{2} \quad \text{.................. (i)}$$

Again, $x = \left[\frac{\ell}{2} - R\sin\theta\right]$

Now, to find out the radius of circular arch.

$$R = \frac{l^2}{8h} + \frac{h}{2}$$

$$= \frac{30^2}{8 \times 6} + \frac{6}{2}$$

$$= 21.75\ m$$

To find angle θ,

we have, $x = (\frac{1}{2} - R\sin\theta)$

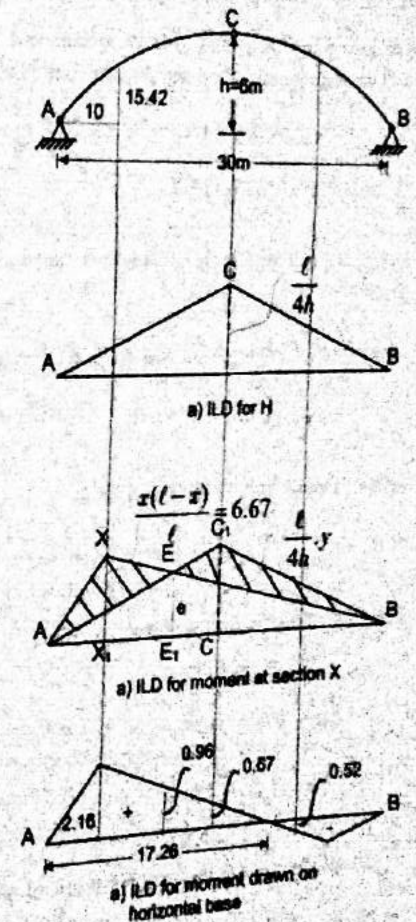


Fig. 6.28

When $x = 10\ m$

$$10 = [15 - 21.75 \sin\theta]$$

or, $10 = 15 - 21.75 \sin\theta$

or, $21.75 \sin\theta = 15 - 10$

$$\therefore\ \theta = \sin^{-1}\left(\frac{5}{21.75}\right) = 13.29^{\circ}$$

Now, rise of span (y) when $x = 10\ m$

$$y = h - R(1 - \cos\theta)$$
$$= 6 - 21.75(1 - \cos 13.29^{\circ})$$
$$= 5.417\ m$$

For horizontal reaction, use influence line diagram. Horizontal reaction H will be maximum, when the 200 kN load is at C at this point.

$$H_{max} = 200 \times 1.25 + 80 \times 1.25 \times \frac{11}{15}$$
$$= 323.33\ kN$$

Let, point E at which ILD ordinate is zero be at a distance x from the left hand support.

Now, from triangles AG_1C and AE_1E, we have,

$$\frac{\ell}{x} = \frac{6.77}{15} \quad \text{..........................} \quad \text{(i)}$$

Again, from triangles BX_1X and BE_1E

$$\frac{\ell}{30 - x} = \frac{6.77}{20} \quad \text{..........................} \quad \text{(ii)}$$

From (i) and (ii) eliminating ℓ,

$$\frac{30 - x}{x} = 0.7377$$

or, $30 = 1.7377\,x$

$$\text{or, } x = \frac{30}{1.7377}$$
$$= 17.26\,m$$

From influence line diagram for moment drawn on horizontal base.

a) Positive maximum bending moment

$$= 200 \times 2.156 + 80 \times 0.965$$
$$= 508.4\ kN$$

b) Maximum Negative bending moment

Similarly, maximum negative moment will occur when 200 kN load is at hinge C, and 80 kN load is to its right. For the load position indicated.

$$M_{max} = 200 \times 0.674 - 80 \times 0.52$$
$$= 93.4\ kN\text{-}m$$

Example # 6.21 A three hinged parabolic arch has a span 50 m and central rise 8 m. Determine the maximum positive and negative bending moment at a section 15 m from left hand support, if a uniformly distributed load of 50 kN/m rolls over the arch. Also, determine the absolute maximum bending moment.

Sol^n.

Given,

span $\quad \ell = 50\ m$

Central rise $= 8\ m$

Distance between the section and left hand support,

$$x = 15\ m$$

Load, $\quad w = 50\ kN/m$

a) Maximum positive bending moment

Let, M_{max} = Maximum positive bending moment at the section.

Using relation

$$M_{max} = \frac{wx(\ell - x)(\ell - 2x)}{2(3\ell - 2x)}$$
$$= \frac{50 \times 15(50 - 15)(50 - 2 \times 15)}{2(3 \times 50 - 2 \times 15)}$$
$$= 2187.5\ kN\text{-}m$$

b) Maximum negative bending moment

We know that maximum negative bending moment is equal to that of the maximum positive bending moment.

$$M_{max} = 2187.5\ kN\text{-}m$$

c) Absolute maximum bending moment

M_{abs} = Absolute maximum bending moment. Then using relation

6.9 Exercise

Ex.1 Draw *BM*, *SF* and Thrust diagrams of the following structures,

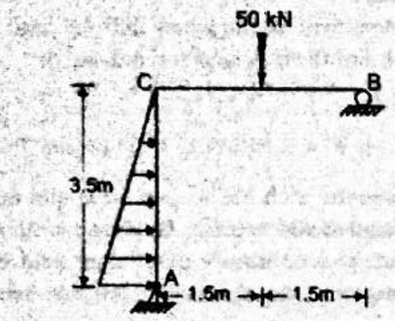


Fig. 6.29

Ans. $R_{AX} = 61.25 \rightarrow$

$R_{Ay} = 11.3\ kN$

$R_{By} = 36.3\ kN \uparrow$

$F_C = +11.3$

$M_C = +71.45$

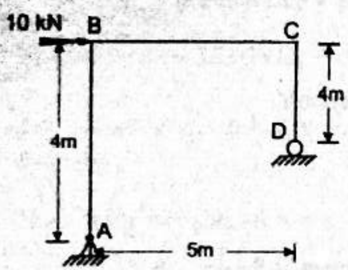


Fig. 6.30

Ans. $R_{Ay} = 8\ kN \downarrow$

$R_{AX} = 10\ kN \leftarrow$

$\underline{AB}$ $T = 8,\ F = -10,\ M = 0, 40$

$\underline{BC}$ $T = 0,\ F = 8,\ M = 0,$ at $C,\ M = 40$ at B

$\underline{CD}$ $T = -8\,kN,\ F = 0,\ M = 0$

Ex. 2 A three-hinged parabolic arch has a span of 125 *m* and a rise of 25 *m*. It is loaded with a uniformly distributed load of 30 *kN/m* covering the central half span.
Determine.
(i) The maximum sagging and hogging moments
(ii) The thrust, shear and moments at the quarter
($x = 27.6\ m$ from *A*, $M = 4767.5\ m$, At right quarter span,
(Ans: $M = -6225.5\ kN\text{-}m$, $V = 937.5\ kN$, $F = 41.6\ kN$, M = 3261 *kN-m*)

Ex. 3 A three-hinged parabolic arch has a span of 50 *m* and is supported at different levels as shown in the figure. Find bending moment at a section 15 *m* form left support when a 100 *kN* vertical load is at this section.

Ex. 4 A three hinged parabolic arch has a span of 75 *m* has is supported at different levels as shown in figure. It carries a uniform load of 15 *kN/m* over the heft half portion. Determine.
(i) The thrust, shear and moment at 15 *m* from the ends.
(ii) The magnitude and position of maximum sagging and hogging moment in the arch.

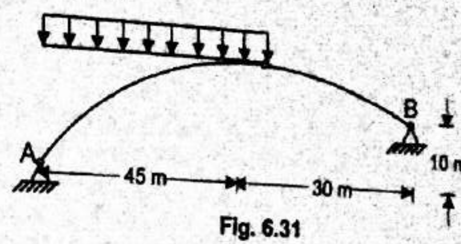


Fig. 6.31

Ex. 5 A symmetrical three-hinged parabolic arch has a span of 30 *m* and central rise of 6 *m*. The arch carries a distributed load, which varies uniformly from 40 *kN/m* at each abutment to zero of mid span.
Determine
(a) The horizontal thrust at abutment.
(b) Maximum positive bending moment in the arch.

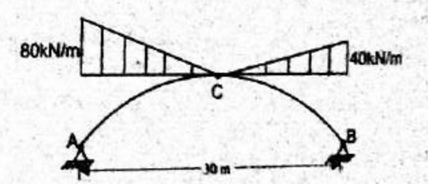


Fig. 6.32

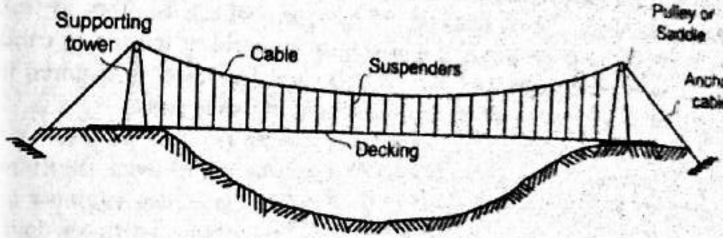
(Ans: $H = 250\ kN$, at $x = 5$m from crown on either side, $M_{max} = 55.55\ kN\text{-}m$)

7

SUSPENSION CABLE SYSTEM

7.1 ELEMENTS OF SUSPENSION BRIDGE

Suspension bridges are used for roadways when the span exceeds 200 m. They consist of the following elements: i) Cable ii) Suspenders iii) Decking iv) Supporting tower v) Anchorage vi) Windguy and Windties. A typical suspension bridge and its components are shown in Fig. (7.1).

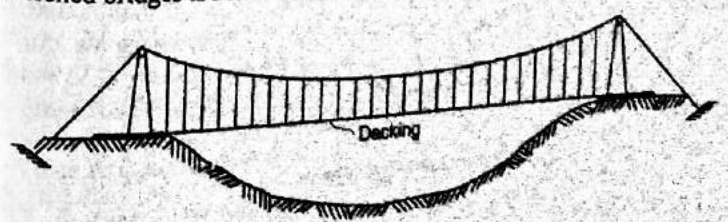


Suspension Bridge

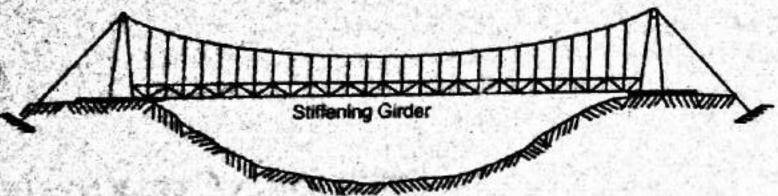
Fig. 7.1

The dead load of the bridge and the traffic load on the deck are carried by cables through suspenders. The cables in general pass over towers and anchored to a proper foundation.

As the cables are flexible members, their curvature changes under the application of load. This is apparent in the case of unstiffened bridges. However, stiffened bridges do not have such problem. The stiffened girder transfers a uniformly distributed load to each suspender. The stiffened and unstiffened bridges are shown below.



a) Unstiffened Suspension Bridge



a) Stiffened Suspension Bridge

Fig. 7.2

Suspension bridges are often supported on towers. Towers provide means for the cables to change the direction. Normally, three types of towers are used in Suspension bridge system.

i) <u>Towers with saddles:</u> Such types of tower are fixed at their bases and supports the main cables through a "carriage" which is free to roll horizontally on the tower top. The horizontal component of tension in cable and tension in backstay must be equal in this case. The tower is required to offer resistance only to the vertical components of the cable pulls.

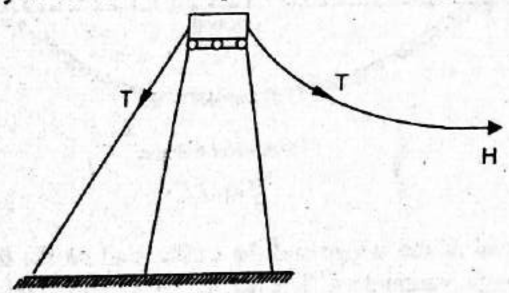


Fig. 7.3

ii) <u>Tower with pulleys at top:</u> Such type of towers act like vertical cantilevers and offer some resistance to cable movement. The towers are subjected to horizontal and vertical forces.

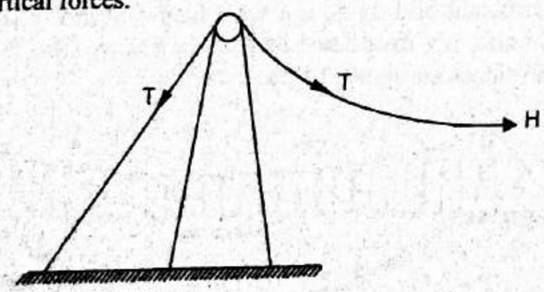


Fig. 7.4

iii) <u>Towers hinged at base:</u> Such type of towers are free to rock in the plane of the main cables, which are securely attached to the tower tops. The towers themselves are thus simple struts between the cables and the tower foundation and offer no resistance to bending moment and the movement of the cables.

Suspension bridges are necessary to be protected from the vibration caused by wind. The lateral stability of the bridge is thus achieved by windguy arrangement. A standard form of windguy arrangement is shown in the figure. It essentially consists of the bridge with a cable in the lateral direction with parabolic shape.

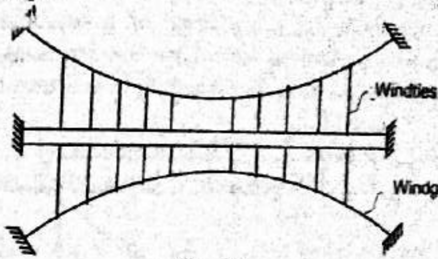


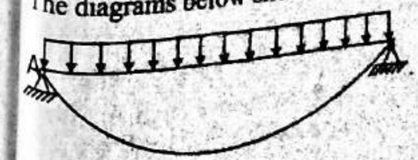
Fig. 7.5

In general the windguy arrangement is placed below the walkway, as this will contribute most to the serviceability of the bridge. The connection between the windguy cable and the walkway of the bridge is made by other cables, which are called windties.

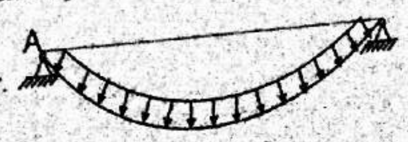
Practically, the windguy cable forms a three dimensional curve. However, for the analysis purpose, it is assumed that it is of parabolic form in plan and side elevation.

7.2 CABLES

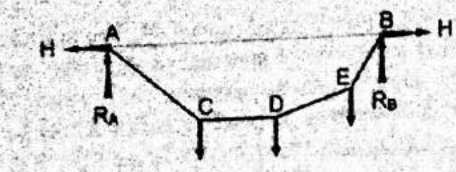
As described before, Cables are the main members of a suspension bridge carrying load of the whole bridge. They are one-dimensional structures and are suspended between supports to carry vertical loads. Since cables are flexible, they cannot resist bending moment and compression forces. As the load is applied, the cable changes its position and the load is carried though the axial tension. If the load in the cable is uniformly distributed along the horizontal span, the shape of the cable will be parabolic and if the load is uniform along the length of the cable, the shape will be catenary. When a number of forces are applied, the cable takes the shape of funicular polygon. The diagrams below show the different shapes of cable.



Parabola



Catenary



Funnicular Polygon

Fig. 7.6

7.3 Equation of a Cable

As mentioned earlier, when a cable is loaded with uniformly distributed load along its span, the cable takes the shape of a parabola. Equations of equilibrium are first applied to determine the reactions and then the derivation of the general equation for cable follows as illustrated below.

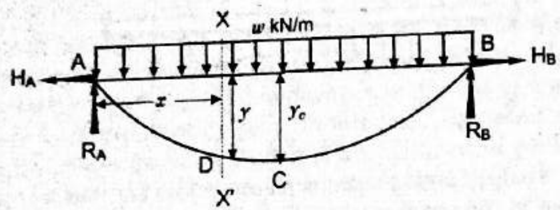


Fig. 7.7

Let, ACB represents a cable loaded with uniformly distributed load of w per unit length.

Applying $\Sigma F_x = 0$, we get, $H_A = H_B = H$

$\Sigma F_y = 0$, $R_A = R_B = \dfrac{w\ell}{2}$ (since load is symmetrical)

$\Sigma M_c = 0$, $H_A \times y_c + w\left(\dfrac{\ell}{2}\right)^2 \cdot \dfrac{1}{2} - R_A \cdot \dfrac{\ell}{2} = 0$

or, $H_A = \dfrac{1}{y_c}\left(\dfrac{w\ell}{2} \cdot \dfrac{\ell}{2} - \dfrac{w\ell^2}{8}\right)$ (Substituting $R_A = \dfrac{w\ell}{2}$)

$\therefore \quad H_A = \dfrac{w\ell^2}{8y_c} = H_B = H$

$$\therefore \quad H = \frac{w\ell^2}{8y_c} \qquad \text{.......... (7.1)}$$

This is the equation for horizontal thrust on the cable. Now to find the equation of the cable, consider a section XX' at a distance x from A. Since bending moment is equal to zero at every pointing a cable,

$\Sigma M_D = 0$

or, $H_A \cdot y + \dfrac{wx^2}{2} - R_A x = 0$

or, $\dfrac{w\ell^2}{8y_c} \cdot y + \dfrac{wx^2}{2} - \dfrac{w\ell}{2} x = 0$ $\qquad \left[\because H_A = \dfrac{w\ell^2}{8y_c}\right]$

or, $\dfrac{w\ell^2}{8y_c} \cdot y = \dfrac{w\ell x}{2} - \dfrac{wx^2}{2}$

or, $\dfrac{w\ell^2}{8y_c} \cdot y = \dfrac{wx}{2}(\ell - x)$

$$\therefore \quad y = \frac{4y_c x}{\ell^2}(\ell - x) \qquad \text{.......... (7.2)}$$

This is the equation for a cable with uniformly distributed load on it. (Note that this equation is similar to that of an arch.)

Now tensions in the cable is obviously the resultant of the reaction forces at the supports, and are

$$T_A = \sqrt{R_A^2 + H_A^2}$$

$$T_B = \sqrt{R_B^2 + H_B^2}$$

The tension forces are shown below

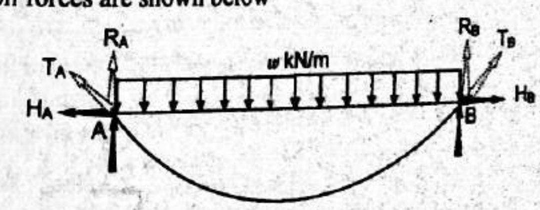


Fig. 7.8

Alternatively, the equation of the cable can also be found out as follows. Equation of the parabola with A as origin is

$$y = kx(\ell - x) \qquad \text{.......... (7.3)}$$

at $x = \dfrac{\ell}{2}$, $y = y_c$

Substituting these in Eq. (7.3), we get

$y_c = k\dfrac{\ell}{2}(\ell - \dfrac{\ell}{2}) = \dfrac{k\ell^2}{4}$

or, $k = \dfrac{4y_c}{\ell^2}$

Now substituting the value of k in Eq. (7.3), we get,

$$y = \frac{4y_c}{\ell^2} x(\ell - x) \qquad \text{(Same as above)}$$

7.4 Equilibrium of Cable

A cable is a flexible structure, which cannot resist bending moment. It deflects so that the bending moment is zero at any point, which is achieved by developing horizontal thrust at the support and thus developing appropriate deflection.

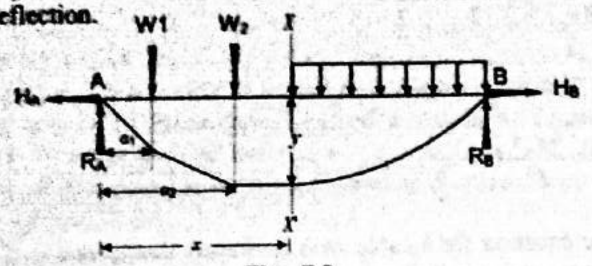


Fig. 7.9

Consider the cable shown in Fig. (7.9), which is subjected to a number of loads. Let the horizontal force developed be H and let R_A and R_B be the vertical reactions at supports A and B. At section X - X', let the deflection be y. Then moment at x is given by

$$M_x = R_A x - W_1(x - a_1) - W_2(x - a_2) - Hy$$

Since the cable is flexible, $M_x = 0$

$$Hy = R_A x - W_1(x - a_1) - W_2(x - a_2)$$

$$\therefore \quad Hy = M_{beam} \text{ (Simply supported beam moment)} \quad \text{.........(7.4)}$$

Using the above equation, a loaded cable can be analyzed.

7.5 Cable Subjected to Concentrated Loads

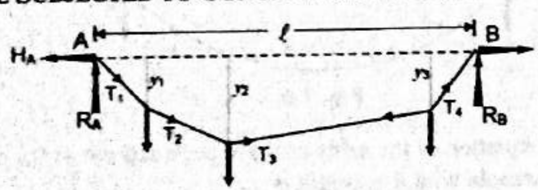


Fig. 7.10

Consider the cable spanning over a horizontal gap ℓ subjected to the concentrated loads as shown in Fig. (7.10). Let R_A and R_B be the vertical and let H $(H_A = H_B = H)$ be the horizontal reactions at supports. The equilibrium condition as given by Eq. (7.4) is

$$Hy = M_{beam}$$

$$\text{or,} \quad y = \frac{M_{beam}}{H}$$

The deflected shape is thus similar to the beam moment diagram. If M_1, M_2 and M_3 are the beam moments at load points 1, 2 and 3, the deflections y_1, y_2 and y_3 are given by.

$$y_1 = \frac{M_1}{H}; \quad y_2 = \frac{M_2}{H}; \quad y_3 = \frac{M_3}{H}$$

If the horizontal thrust is known or position of cable at any one point is known, the deflections at all the other points can be computed and the shape of the cable can be determined.

Example # 7.1 A light cable 18 *m* long is supported at two ends at the same level. The supports are 16 *m* apart. The cable supports 120 *kN* dividing the distance into two equal parts. Find the tension in the cable

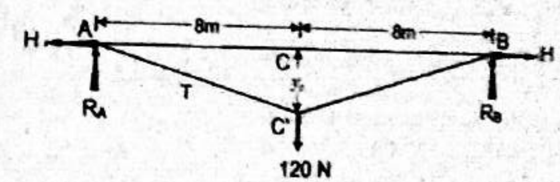


Fig. 7.11

Referring to the Fig. (7.11), let y_c be the deflection at the centre C. By symmetry,

$$R_A = R_B = \frac{120}{2} = 60 \text{ kN}$$

Beam moment at C, $M_c = R_B \times 8 = 60 \times 8 = 480$ *kN-m*. Let H be the horizontal reaction. Then,

For equilibrium of point C',

$$H.y_c = M_c \qquad \left(\text{But } y_c = \sqrt{9^2 - 8^2} = 4.123 \text{ m}\right)$$

$$\text{or,} \quad H = \frac{480}{4.123} = 116.42 \text{ N.}$$

Now the tension in the cable

$$= \sqrt{R_A^2 + H^2}$$

$$= \sqrt{60^2 + 116.42^2} = 130.97 \text{ kN} \quad \underline{\text{Ans.}}$$

Example # 7.2 A light flexible cable shown in Fig.(7.12) is supported at two ends at the same level and 12 *m* apart. The cable is subjected to two loads of 15 *kN* and 30 *kN* at a distance of 4*m* and 8 *m* from *A*. The dip of the cable under 30 *kN* load is 1.5 *m*. Determine the horizontal component of tension in the cable. Also calculate total length of the cable.

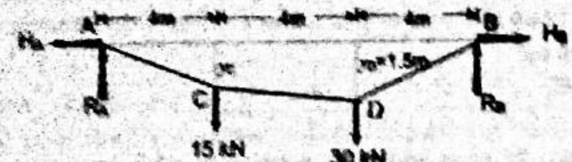


Fig. 7.12

Sol^n. In order to find the horizontal component of tension in cable, apply the general cable theorem at point D.

$$H.y_D = M_D \quad \text{(Beam moment at } D\text{)}$$

or, $$H = \frac{M_D}{y_D} = \frac{R_B \times 4}{1.5}$$

R_B is determined by taking moment about A.

or, $$\sum M_A = 0 = R_B \times 12 - 30 \times 8 - 15 \times 4$$

or, $R_B = 25$

Substituting this in the above expression for H, we get,

$$H = \frac{25 \times 4}{1.5} = 66.67\,kN$$

Again we have,

$$H.y_C = M_C = R_A \times 4$$

$$\therefore y_C = \frac{R_A \times 4}{H} = \frac{20 \times 4}{66.67} = 1.2\,m$$

Now, Length of cable $= \sqrt{4^2 + 1.2^2} + \sqrt{4^2 + (1.5 - 1.2)^2} + \sqrt{4^2 + 1.5^2}$

$= 12.46\ m$ Ans.

Example # 7.3 A cable is used to support six equal and equidistant loads over a span of 49 m. The central dip of the cable is 5 m and loads are 2 tons each. Find the length of the cable required and its sectional area if the safe tensile stress is 1.574 t/cm^2.

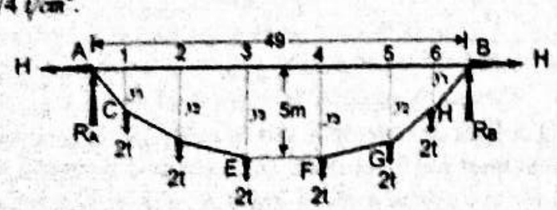


Fig. 7.13

Sol^n.

By symmetry, the dip at different points can be written as y_1, y_2 and y_3 as in Fig. (7.13).

Also, $R_A = R_B = \frac{12}{2} = 6t$

We have, beam moment $M_1 = M_6 = R_A \times 7 = 6 \times 7 = 42$

$$M_2 = M_5 = R_A \times 14 - 2 \times 7 = 6 \times 14 - 14 = 70$$

$$M_3 = M_4 = R_A \times 21 - 2 \times 14 - 2 \times 7$$

$$= 6 \times 21 - 2 \times 14 - 2 \times 7 = 84$$

Now, $Hy_3 = M_3 = 84$

or, $H = \frac{84}{5} = 16.8$

Again, $Hy_1 = M_1 = 42$

or, $y_1 = \frac{42}{16.8} = 2.5\ m$

Similarly,

$Hy_2 = M_2 = 70$

or, $y_2 = \frac{70}{16.8} = 4.17\ m$

Length of cable $= 2(AC + CD + DE) + 7$

$$= 2\left[\sqrt{y_1^2 + 7^2} + \sqrt{(y_2 - y_1)^2 + 7^2} + \sqrt{\{(y_3 - y_1) - (y_2 - y_1)\}^2 + 7^2}\right] + 7$$

$$= 2\left(\sqrt{2.5^2 + 7^2} + \sqrt{(4.17 - 2.5)^2 + 7^2} + \sqrt{\{(5 - 2.5) - (4.17 - 2.5)\}^2 + 7^2}\right) + 7 = 50.3$$

$\ldots$ m Ans.

Tension in the cable $= \sqrt{R_A^{\,2} + H^2}$

$$T = \sqrt{6^2 + 16.8^2} = 17.84t$$

Area $= \frac{T}{\sigma} = \frac{17.84}{1.574} = 11.33\ cm^2$ Ans.

Example # 7.4 A suspension cable is supported at two points A and B, A being one meter above B. The distance AB is equal to 20 m. The cable is subjected to loads of 2 kN, 4 kN, 5 kN and 3 kN at distances of 4 m, 8 m, 12 m and 16 m respectively from A. Find the maximum tension in the cable, if the dip of the cable at point of application of first load is 1m with respect to level at A. Find also the length of the cable.

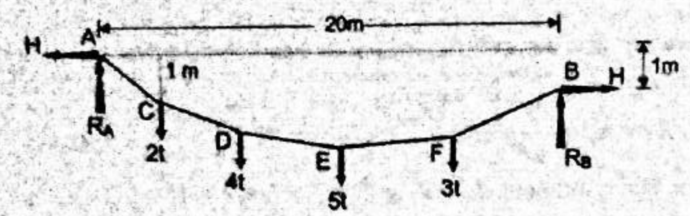


Fig. 7.14

Sol^n. Let R_A and R_B be the vertical reactions at A and B respectively

$\sum F_x = 0, \quad H_A = H_B = H$ ....(i)

$R_A + R_B = 2 + 4 + 3 + 5 = 14\ kN$ ....(ii)

$\sum M_c = 0, \quad H_A \times 1 - R_A \times 4 = 0$

or, $\quad H = 4R_A \quad (H_A = H)$ ....(iii)

$\sum M_A = 0,\ R_B \times 20 + H \times 1 - 3 \times 16 - 5 \times 12 - 4 \times 8 - 2 \times 4 = 0$

$20R_B + H = 148$ ....(iv)

Substituting the value of H in Eq. (iv) we get,

$4R_A + 20R_B = 148$ ....(v)

$R_A + R_B = 14$

Solving (v) and (i), we get

$16R_B = 148 - 14 \times 4$

or, $\quad R_B = 5.75\ kN$

Substituting the value of R_B in Eq. (i), we get,

$R_A = 14\text{-}5.75 = 8.25\ kN$

Substituting the value of R_B in Eq. (iv), $H = 148 - 20 \times 5.75 = 33\ kN$

Since $R_A > R_B$, maximum tension in the cable will occur at A and its value is given by

$$T_{max} = \sqrt{8.25^2 + 33^2} = 34.02\ kN.$$

$\sum M_D = 0$

or, $\quad H \times y_D + 2 \times 4 - R_A \times 8 = 0$

or, $\quad 33 \times y_D + 8 - 8.25 \times 8 = 0$

or, $\quad y_D = 1.76\ m$

$\sum M_E = 0$

$H.y_E + 2 \times 8 + 4 \times 4 - R_A \times 12 = 0$

or, $\quad y_E = 2.03$

$\sum M_F = 0$

$H.y_F + 2 \times 12 + 4 \times 8 + 5 \times 4 - R_A \times 16 = 0$

or, $\quad 33y_F + 24 + 32 + 20 - 8.25 \times 16 = 0$

or, $\quad y_F = 1.7\ m$

Now, the lengths of the different portions are,

$$AC = \sqrt{4^2 + 1^2} = 4.123m.$$

$$CD = \sqrt{4^2 + (1.76 - 1)^2} = 4.07m.$$

$$DE = \sqrt{4^2 + (2.03 - 1.76)^2} = 4.01m.$$

$$EF = \sqrt{4^2 + (2.03 - 1.76)^2} = 4.01m.$$

$$FB = \sqrt{4^2 + (1.7 - 1)^2} = 4.06m.$$

Total length of the cable $= AC + CD + DE + EF + FB$

$= 4.123 + 4.07 + 4.01 + 4.01 + 4.06 = 20.273\ m$ Ans.

Example # 7.5 A cable has to be suspended between two points such that its central dip is 1/15 th of the span. If the cable is of uniform cross section and its unit weight is 7938 kg/m^3, calculate the length of the cable. Safe permissible tensile stress in the material is 1.26 t/cm^2

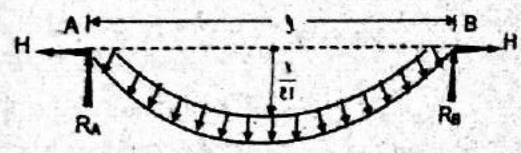


Fig. 7.15

Given,

Density of the cable $= 7938\ kg/m^3$

Safe permissible stress $= 1.26\ t/cm^2$

Let w be the load per unit horizontal span of the cable, then

$$R_A = \frac{w\ell}{2} = 0.5w\ell$$

and $\quad H = \dfrac{w\ell^2}{8y_c} = \dfrac{w\ell^2}{8 \times \frac{\ell}{15}} = 1.875w\ell$

Now, $T = \sqrt{R_A^2 + H^2} = \sqrt{(0.5w\ell)^2 + (1.875w\ell)^2} = 1.94w\ell$

We have, $\quad \dfrac{T}{a} = \sigma$

or, $\quad \dfrac{1.94w\ell}{a} = 1.26 \times 1000 \times 100 \times 100$ ....(i)

a = Area

S = Length of cable

$$w = \frac{\text{total cable load}}{\text{span}} = \frac{7938 \times a.S}{\ell}$$

Substituting the value of w in Eq. (i), we get

$$\frac{1.94\ell}{a} \cdot \frac{7938a \times S}{\ell} = 12600000$$

or, $S = 818.19\ m$ Ans.

Example # 7.6 A foot bridge of width 3 *m* and span 50 *m* is carried by two cables of uniform section having a central dip of 5 *m*. If the platform load is 5 kN/m^2, calculate the maximum pull in the cables. Find the necessary sectional area required for the cables if the allowable stress is 120 N/mm^2

Sol^n. The cable is shown in Fig. (7.16)

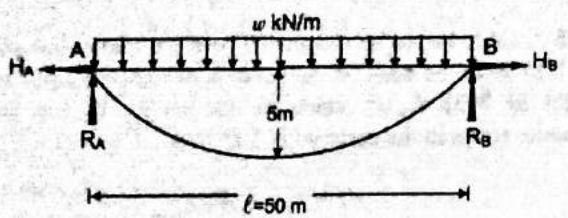


Fig. 7.16

Area of platform = 3 × 50 = 150 m^2
Total load of the bridge = 150 × 5 = 750 *kN*
Since the load is carried by two cables,

Load in each cable $= \frac{750}{2} = 375\ kN$

Load per unit meter $= w = \frac{load}{span} = \frac{375}{50} = 7.5\ kN/m$

Now, $R_A = R_B = \frac{w\ell}{2} = \frac{7.5 \times 50}{2} = 187.5\ kN$

$$H = H_A = H_B = \frac{w\ell^2}{8y_c} = \frac{7.5 \times 50^2}{8 \times 5} = 468.75\ kN$$

The pull at *A* in the cable

$$T_A = \sqrt{R_A^2 + H_A^2} = \sqrt{187.5^2 + 468.75^2} = 504.86\ kN$$

Similarly $T_B = 504.86\ kN$
Now, the sectional area of the cable is calculated by the expression

$$stress = \frac{Force}{Area}$$

$$Area = \frac{Force}{Stress} = \frac{504.86}{120}$$

$$= 4207.17\ mm^2 \quad \text{Ans.}$$

Example # 7.7 A footbridge of width 3 *m* and span 60 *m* is carried by two cables of uniform section having central dip of 5 *m*. If the platform load is 4 kN/m^2, calculate the maximum pull in each cable due to this load. What will be the central dip in each cable if a single concentrated load of amount equal to total platform load as above acts at the centre of each cable instead of having bridge? Neglect self-weight of the cables.

[T.U. 2058 Jestha]

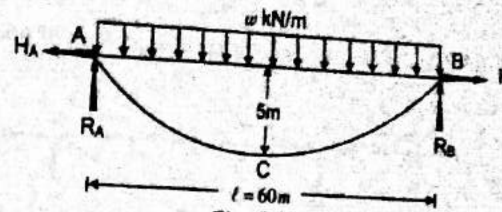


Fig. 7.17

Sol^n.

Span $\ell = 60\ m$
Central dip $y_c = 5\ m$
Width of footbridge = 3 *m*
Platform load $= 4\ kN/m^2$

∴ Total uniformly distributed load carried by each cable

$$w = \frac{3 \times 4}{2}\ kN/m = 6\ kN/m$$

Let, R_A = Vertical reaction at *A*, R_B = Vertical reaction at *B*
H = Horizontal thrust.

∴ $R_A = \frac{w\ell}{2} = 6 \times \frac{60}{2} = 180\ kN$

Now, $M_C = R_A \times \frac{\ell}{2} - w.\frac{\ell}{2} \times \frac{\ell}{4}$

$$= 180 \times 30 - 6 \times \frac{60}{2} \times \frac{60}{4}$$

$$= 5400 - 2700$$

$$= 2700\ kN-m$$

Now, horizontal thrust $H = \frac{M_C}{y_c} = \frac{2700}{5} = 540\ kN$

∴ Max. Pull $T_{max} = \sqrt{R_A^2 + H^2}$

$$= \sqrt{180^2 + 540^2}$$

$$= 569.20\ kN \quad \text{Ans.}$$

Again, the single concentrated load equivalent to uniformly distributed load on each cable is

$$= 6\ kN/m \times 60\ m = 360\ kN$$

Now, length of the cable

$$= \ell + \frac{8}{3} \times \frac{y_c^2}{\ell} = 60 + \frac{8}{3} \times \frac{5^2}{60} = 61.11\ m$$

Since the length of the cable is same when an equal amount of point load acts at the center,

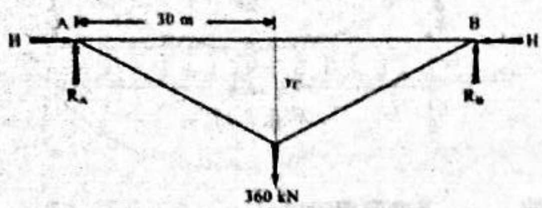


Fig. 7.18

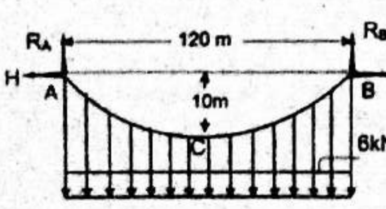
$\therefore \quad S = 2 \times \sqrt{(30^2 + y_c^2)}$

or, $\quad 61.11 = 2.\sqrt{30^2 + y_c^2}$

or, $\quad 30.55^2 = 30^2 + y_c^2$

or, $\quad y_c = 5.79\ m$ Ans.

Example # 7.8 A cable of span 120 *m* and dip 10 *m* carries a load of 6 *kN/m* along the horizontal span. Find the maximum tension in the cable at the support. Also, find the forces transmitted to the supporting pier if the cable "passes over smooth pulleys on top of the pier. The anchor cable is at 30^0 to the horizontal. Determine the maximum bending moment for the pier if the height of the pier is 15 *m*.



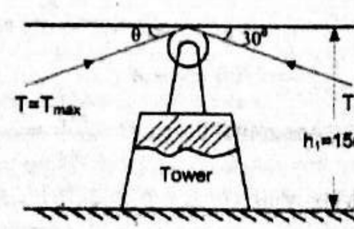


Fig. 7.19

Sol^n. Referring to Fig. (7.19 – a)
Due to symmetry,

$$R_A = R_B = \frac{w\ell}{2} = \frac{6 \times 120}{2} = 360\ kN$$

Since, $\quad H = \frac{w\ell^2}{8y_c} = \frac{6 \times 120^2}{8 \times 10} = 1080\ kN$

$$T_{max} = \sqrt{R^2 + H^2} = \sqrt{360^2 + 1080^2}$$

$$= 1138.42\ kN \text{ Ans.}$$

$$\cos\theta = \frac{H}{T_{max}} = \frac{1080}{1138.42}$$

$$\theta = 18.435^0 \text{ Ans.}$$

Referring to Fig. (7.19 – b), horizontal force transferred to pier

$= T_{max}(\cos\theta - \cos\alpha)$

$= T_{max}(\cos 18.435^0 - \cos 30^0)$

$= 1138.42(\cos 18.435^0 - \cos 30^0)$

$= 94.099\ kN$ Ans.

Max. bending moment in the pier

$= Hh_1$

$= 94.099 \times 15$

$= 1411.49\ kN\text{-}m$ Ans.

Vertical force on the pier

$= T_{max}(\sin\theta + \sin\alpha)$

$= 1138.42(\sin 18.435^0 + \sin 30^0)$

$= 929.21\ kN$ Ans.

Example # 7.9 A uniform flexible cable of weight *w* per unit length connects two points, which are at the same level. Prove that difference between maximum and minimum tension of a cable is $w.y_c$.

Sol^n.

Let the tension in cable is *T*. Then,

$T\cos\theta = H$

$T = H\sec\theta = H\sqrt{1 + \tan^2\theta}$

$$T = H\sqrt{1+\left(\frac{dy}{dx}\right)^2} \text{ and } H = \frac{w\ell^2}{8y_c}$$

The equation of cable with the origin at left support is,

$$y = \frac{4y_c\, x(\ell - x)}{\ell^2} \text{ and } \frac{dy}{dx} = \frac{4y_c(\ell - 2x)}{\ell^2}$$

The maximum tension will occur at the end $x = 0$

$$T_{max} = \frac{w\ell^2}{8y_c}\sqrt{1+\frac{16y_c^{\,2}}{\ell^2}}$$

The Minimum tension in the cable will occur at centre,

i.e. $x = \frac{\ell}{2}$, $\therefore T_{min} = \frac{w\ell^2}{8y_c}$

The difference between maximum and minimum tension

$$= T_{max} - T_{min}$$

$$= \frac{w\ell^2}{8y_c}\left[\sqrt{1+\frac{16y_c^{\,2}}{\ell^2}} - 1\right]$$

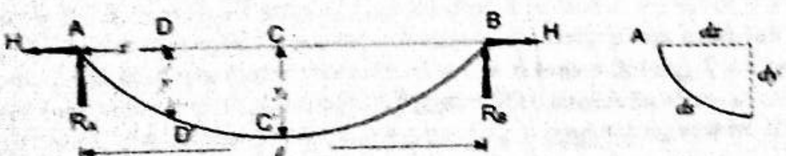
$$= \frac{w\ell^2}{8y_c}\left[1+\frac{8y_c^{\,2}}{\ell^2}\text{.......} - 1\right] \qquad \left[\text{When } \frac{y_c}{\ell} < 1\right]$$

$$= w.y_c$$

7.6 Length of Cable

a) When both the ends of the cables are at same level.

Let ADCB be the parabolic cable whose length is to be determined.



(a) (b)

Fig. 7.20

Consider small length ds at end A of the cable, Fig. (7.20-b), we have ,

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$= \left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx \text{ .... (7.5)}$$

The terms dy/dx in the above equation is obtained by differentiating the cable equation. The general equation of a parabola with its origin at C is

$$y = kx^2 \text{ .... (7.6)}$$

At $x = \frac{\ell}{2}$, $y = y_c$, substituting these in Eq. (7.6) we get

$$k = \frac{y}{x^2} = \frac{y_c}{\left(\frac{\ell}{2}\right)^2} = \frac{4y_c}{\ell^2}$$

Substituting the value of k in Eq. (7.6) we get,

$$y = \frac{4y_c}{\ell^2}x^2$$

or, $$\frac{dy}{dx} = \frac{8y_c}{\ell^2}x$$

Substituting the value of dy/dx in Eq. (7.5),

$$ds = \left[1+\left(\frac{8y_c}{\ell^2}.x\right)^2\right]^{\frac{1}{2}} dx = \left(1+\frac{64y_c^2.x^2}{\ell^4}\right)^{\frac{1}{2}} dx$$

Expanding the right hand term using Binomial expansion, we get,

$$ds = \left[1+\frac{1}{2}.\frac{64y_c^2x^2}{\ell^4}+\frac{1}{2}\frac{\left(1-\frac{1}{2}\right)}{2!}\left(\frac{64y_c^2x^2}{\ell^4}\right)^3+\text{.....}\right].dx$$

Neglecting the terms of higher powers of $\frac{y_c^2.x^2}{\ell^4}$, we get

$$ds = \left(1+\frac{32y_c^2.x^2}{\ell^4}\right)dx$$

Now, the total length is given by

$$S = 2\int_0^{\ell/2} ds$$

$$= 2\int_0^{\ell/2}\left(1+\frac{32y_c^2.x^2}{\ell^4}\right)dx$$

$$= 2\left[\frac{\ell}{2}+\frac{32y_c^2}{\ell^4}.\frac{\ell^3}{8}\times\frac{1}{3}\right]$$

$$S = \ell + \frac{8}{3}.\frac{y_c^2}{\ell}$$

This is the expression for finding the length of cable when the supports are at the same level.

b) When the ends of the cables are at different level.

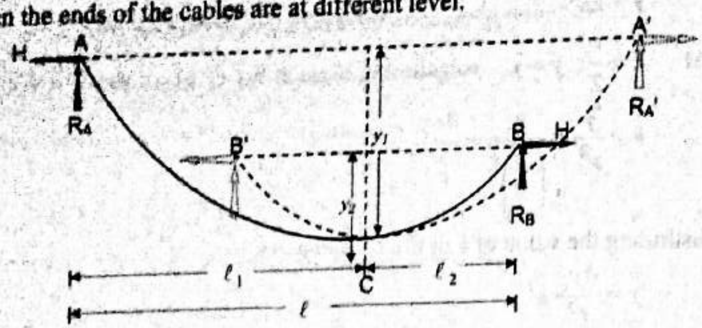


Fig. 7.21

Consider a cable ACB supported at different levels A and B and C is the lowest. The point C is extended to A' and B' as shown by the dotted lines such that $AC = A'C$ and $CB = CB'$. By the extension of the curves, level of A' and B' are same as of A and B respectively. Now as seen in the Fig. (7.21), we have two cables ACA' with the dip y_1 and BCB' with the dip of y_2.

Now, we have

$$\ell_1 + \ell_2 = \ell \quad \text{.......... (7.7)}$$

Considering AA'

$$H = \frac{w\ell^2}{8y_2} = \frac{w(2\ell_1)^2}{8y_1} = \frac{w\ell_1^2}{2y_1} \quad \text{.......... (7.8)}$$

Considering BB'

$$H = \frac{w(2\ell_2)^2}{8y_2} = \frac{w\ell_2^2}{2y_2} \quad \text{..........(7.9)}$$

Equating the Eq. (7.8) and (7.9),

$$\frac{\ell_1^2}{\ell_2^2} = \frac{y_1}{y_2}$$

$$\therefore \quad \frac{\ell_1}{\ell_2} = \sqrt{\frac{y_1}{y_2}} \quad \text{.......... (7.10)}$$

Now, from previous theorem,

Length of the cable $ACA' = S_1 = 2\ell_1 + \frac{8}{3}\frac{y_1^2}{2\ell_1} = 2\ell_1 + \frac{4}{3}\frac{y_1^2}{\ell_1}$

Length of the cable $B'CB' = S_2 = 2\ell_2 + \frac{8}{3}\frac{y_2^2}{2\ell_2} = 2\ell_2 + \frac{4}{3}\frac{y_2^2}{\ell_2}$

The actual length of the cable $S = \frac{S_1 + S_2}{2}$

$$= \frac{1}{2}\left\{\left(2\ell_1 + \frac{4}{3}\frac{y_1^2}{\ell_1}\right) + \left(2\ell_2 + \frac{4}{3}\frac{y_2^2}{\ell_2}\right)\right\}$$

$$= \ell_1 + \frac{2y_1^2}{3\ell_1} + \ell_2 + \frac{2y_2^2}{3\ell_2}$$

But $\ell_1 + \ell_2 = \ell$

$$\therefore \quad S = \ell + \frac{2y_1^2}{3\ell_1} + \frac{2y_2^2}{3\ell_2}$$

This is the expression for the length of cable supported at different levels. The reactions R_A and R_B in the cable can be found out by simply taking moment about B and A respectively. However, one can derive the following expression, to be used directly in solving problems. Let the cable carries uniformly distributed load of w

$$\Sigma M_B = 0$$

$$R_A = \frac{1}{\ell}\left\{\frac{w\ell^2}{2} + H(y_1 - y_2)\right\} \qquad \text{but, } H = \frac{w\ell_1^2}{2y_1}$$

$$= \frac{1}{\ell}\left\{\frac{w\ell^2}{2} + \frac{w\ell_1^2}{2y_1}(y_1 - y_2)\right\}$$

$$= \frac{w}{2\ell}\left(\ell^2 + \frac{\ell_1^2}{y_1}.y_1 - \frac{\ell_1^2}{y_1}.y_2\right)$$

$$= \frac{w}{2\ell}\left(\ell^2 + \ell_1^2 - \ell_1^2 \times \frac{\ell_2^2}{\ell_1^2}\right) \qquad \because \sqrt{\frac{y_2}{y_1}} = \frac{\ell_2}{\ell_1}, \ Eq(7.10)$$

$$= \frac{w}{2\ell}\left(\ell_1^2 + 2\ell_1\ell_2 + \ell_2^2 + \ell_1^2 - \ell_2^2\right) \qquad \because \ell_1 + \ell_2 = \ell$$

$$= \frac{w}{2\ell}\left(2\ell_1^2 + 2\ell_1\ell_2\right) = \frac{2\ell_1 w}{2\ell}(\ell_1 + \ell_2)$$

$$= \frac{2w\ell_1}{2\ell}.\ell = w\ell_1$$

$$\Sigma Fy = 0, \quad R_A + R_B = w\ell$$

or, $\quad R_B = w\ell - w\ell_1 = w(\ell - \ell_1)$

$$= w(\ell_1 + \ell_2 - \ell_2) = w\ell_2$$

Thus for the reactions R_A and R_B, one may use the following expressions directly

$$R_A = w\ell_1 \qquad \text{and} \qquad R_B = w\ell_2$$

Example # 7.10 Derive an expression for shape and length of a catenary cable under its self weight.

Sol^n. When cable is suspended from two supports, it takes shape of catenary under self-weight. Consider a cable suspended from two supports. Let w be self- weight of cable per unit length.

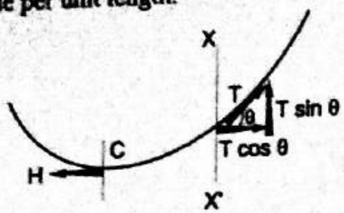


Fig. 7.22

Let H be the tension in cable at the lowest point and T be tension in cable at some section X-X'. Let θ be the slope of cable at the section. Considering equilibrium of right portion of the cable.

$$H = T\cos\theta \qquad \text{(i)}$$

$$wS = T\sin\theta \qquad \text{(ii)}$$

Where S is the length of cable between lowest point and at section X.

From (i),

$$T = \frac{H}{\cos\theta}$$

Substituting this in Eq. (ii), we get,

$$\text{or, } wS = \frac{H}{\cos\theta}\times\sin\theta \qquad \text{(iii)}$$

We have, $S = \int_0^x ds = \int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \because ds = \sqrt{dx^2+dy^2}$

Substituting this in Eq. (iii), we get,

$$w\int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,.dx = \frac{H\sin\theta}{\cos\theta} = H\tan\theta$$

But, $\tan\theta = \frac{dy}{dx}$

$$\therefore \quad w\int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = H\frac{dy}{dx}$$

Differentiating both sides with respect to x,

$$w\sqrt{1+\left(\frac{dy}{dx}\right)^2} = H\frac{d^2y}{dx^2}$$

Put $\frac{dy}{dx} = \theta$ and $\frac{d^2y}{dx^2} = \frac{d\theta}{dx}$

$$\therefore \quad \sqrt{1+\theta^2} = \frac{H}{w}\frac{d\theta}{dx}$$

$$\text{or, } \quad \frac{d\theta}{\sqrt{1+\theta^2}} = \frac{w}{H}dx$$

Integrating both sides.

$$\int\frac{d\theta}{\sqrt{1+\theta^2}} = \frac{w}{H}\int dx$$

$$\log_e\left(\theta+\sqrt{1+\theta^2}\right) = \frac{wx}{H}$$

$$\therefore \quad \theta+\sqrt{1+\theta^2} = e^{\frac{wx}{H}}$$

$$\text{or, } \quad \sqrt{1+\theta^2} = e^{\frac{wx}{H}}-\theta$$

$$\text{or, } \quad 1+\theta^2 = \left(e^{\frac{wx}{H}}-\theta\right)^2$$

$$\text{or, } \quad 1+\theta^2 = e^{\frac{2wx}{H}}+\theta^2-2\theta e^{\frac{wx}{H}}$$

$$\text{or, } \quad 2\theta.e^{\frac{wx}{H}} = e^{\frac{2wx}{H}}-1$$

$$\therefore \quad \theta = \frac{1}{2}\left[e^{\frac{wx}{H}}-e^{-\frac{wx}{H}}\right]$$

$$\text{or, } \quad \frac{dy}{dx} = \frac{1}{2}\left[e^{\frac{wx}{H}}-e^{-\frac{wx}{H}}\right]$$

$$\text{or, } \quad dy = \frac{1}{2}\left[e^{\frac{wx}{H}}-e^{-\frac{wx}{H}}\right]dx$$

egrating,

$$y = \frac{1}{2}\left[e^{\frac{wx}{H}}+e^{-\frac{wx}{H}}\right]\frac{H}{w}$$

$$= \frac{H}{w}\left[\frac{e^{\frac{wx}{H}} + e^{-\frac{wx}{H}}}{2}\right]$$

$$= \frac{H}{w}\cosh\frac{wx}{H}$$

This represents the equation of catenary
Length of cable,

$$S = \int ds = \int \frac{ds}{dx}dx$$

$$= \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^x \sqrt{(1 + \sin h^2 \frac{wx}{H})}\, dx$$

$$= \int_0^x \cos h \frac{wx}{H} dx$$

$$= \frac{H}{w}\sin h\, \frac{wx}{H} + C$$

At $x = 0,\ S = 0,\ \therefore C = 0$

$$\therefore \quad S = \frac{H}{w}\sin h \frac{wx}{H}$$

Example # 7.11 A cable is stretched across a gap of 200 *m* and supports a uniformly distributed load of 15 *kN/m* horizontally. If the central dip is 1/5 *th* of the span, determine the maximum tension in the cable and the length of the cable if the supports are at the same level.

Sol^n. Here, $\ell = 200\ m$

$w = 15 kN/m$

Central dip $y_c = 1/5 \times 200$

$= 40\ m$

As the two supports are at the same level,

$$R_A = \frac{w\ell}{2} = \frac{15 \times 200}{2} = 1500\ kN$$

$$R_B = \frac{w\ell}{2} = 1500\ kN$$

Again, using relation

$$H = \frac{w\ell^2}{8y_c} = \frac{15 \times 200^2}{8 \times 40} = 1875\ kN$$

$\therefore$ Max. Tension $T_{max} = \sqrt{R_A^2 + H^2}$

$= \sqrt{1500^2 + 1875^2}$

$= 2401.17\ kN$ Ans.

Again, length of cable $s = \ell + \frac{8y_c^2}{3\ell}$

$= 200 + \frac{8 \times 40^2}{3 \times 200}$

$= 221.33\ m$ Ans.

Example # 7.12 A steel cable, of uniform section, is hung in the form of parabola between same levels. Find the maximum horizontal span and length of cable if the central dip is 1/12 of the span and the stress in the cable is not to exceed 122 *MPa*. Take density of the steel as 78 kN/m^3.

Sol^n. Given that,

Density of the steel = 78 kN/m^3

Let ℓ = span of parabola in *m*.

$\therefore$ Central dip $= \frac{\ell}{12}$

S = length of cable

Using relation $S = \ell + \frac{8y_c^2}{3\ell}$

$$= \ell + \frac{8(\ell/12)^2}{3\ell} = \frac{55\ell}{54} m$$

We know that total weight of the cable,

$$W = \frac{aS}{100^2} \times 78 \times 100 \quad (\because a = \text{Sectional area of cable})$$

$$= \frac{a}{10000} \times \frac{55\ell}{54} \times 7800 = 0.794 a \times \ell$$

Again we know,

$$H = \frac{wl^2}{8 \times y_c} = \frac{W\ell}{8 \times \frac{\ell}{12}} \quad (\because W = w\ell)$$

$$= \frac{3W}{2}$$

We also know that vertical reaction $R = \frac{w\ell}{2} = \frac{W}{2}$

and the maximum tension in cable $T_{max} = \sqrt{R^2 + H^2}$

$$= \sqrt{\left(\frac{W}{2}\right)^2 + \left(\frac{3W}{2}\right)^2} = 1.58W$$

But, $T_{max} = \sigma \times a$

$1.58W = 1200 \times a$

or, $1.58W \times 0.79 al = 1200 \times a$

$\ell = 955.5\ m$

So the length of the cable,

$$S = \frac{55\ell}{54} = 973.19\ m \ \underline{\text{Ans.}}$$

Example # 7.13 A cable is suspended and loaded as shown in Fig. (7.23).

(a) Compute the length of the cable.
(b) Compute the horizontal component of tension H in cable.
(c) Determine the magnitude and position of maximum tension in the cable.

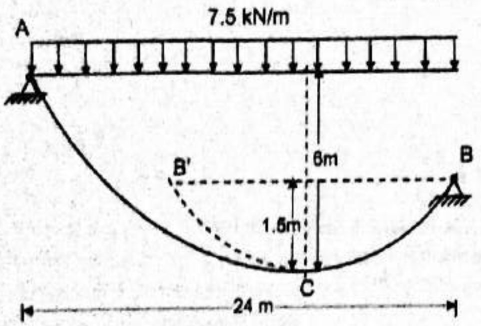


Fig. 7.23

Sol^n.
We have,

$$\frac{\ell_1}{\ell_2} = \sqrt{\frac{y_1}{y_2}} = \sqrt{\frac{6}{1.5}} = 2$$

or, $\ell_1 = 2\ell_2$

But, $\ell_1 + \ell_2 = 24$

or, $2\ell_2 + \ell_2 = 24$

$\therefore \quad \ell_2 = 8$, and $\ell_1 = 16$

(a) Total length

$$S = \ell + \frac{2y_1^2}{3\ell_1} + \frac{2y_2^2}{3\ell_2} = 24 + \frac{2\times 6^2}{3\times 16} + \frac{2\times 1.5^2}{3\times 8} = 25.69 \ \underline{\text{Ans.}}$$

(b) Horizontal Tension, $H = \frac{w\ell_1^2}{2y_1} = \frac{7.5\times 16^2}{2\times 6} = 160\ kN.$

(Alternatively, $H = \frac{w\ell^2}{(2\sqrt{y_1} + \sqrt{y_2})^2}$ may be used.)

(c) $R_A = w\ell_1 = 7.5\times 16 = 120\ kN$. (Alternatively $\sum M_A = 0$ may be used here)

$R_B = w\ell_2 = 7.5\times 8 = 60\ kN < R_A$

$\therefore$ Maximum Tension occurs at support A and is given by

$$T_{max} = \sqrt{R_A^2 + H^2} = \sqrt{120^2 + 160^2}$$

$$= 200\ kN \ \underline{\text{Ans.}}$$

Example # 7.14 A light flexible cable shown below hangs between two points, which are separated horizontally by a distance ℓ. The cable carries a uniformly distributed load, w per unit horizontal length over entire span. If the dip of the lowest point of the cable is y_1 and y_2 from left and right support respectively, show that the horizontal component of tension in the cable is

$$H = \frac{w\ell^2}{2\left(\sqrt{y_1} + \sqrt{y_2}\right)^2}$$

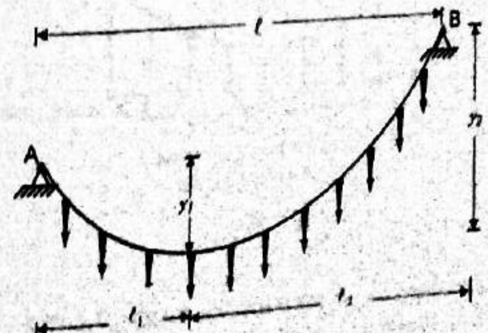


Fig. 7.24

Sol^n. Let ℓ_1 and ℓ_2 be the distance for the lowest point of the cable from left and right support respectively. Then,

$$\ell_1 + \ell_2 = \ell \quad \text{.......... (i)}$$

At the lowest point of the cable, tension in the cable is H. Taking moment about each support,

$$H \times y_1 = \frac{w\ell_1^2}{2} \quad \text{.......... (ii)}$$

and $$H \times y_2 = \frac{w\ell_2^2}{2} \quad \text{.......... (iii)}$$

Eliminating H from Eq. (ii) and (iii)

$$\frac{\ell_1}{\ell_2} = \sqrt{\frac{y_1}{y_2}} \quad \text{.......... (iv)}$$

Solving for ℓ_1 from Eq. (i) and (iv)

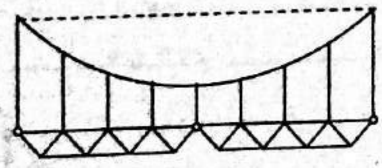
$$\ell_1 = \frac{\ell\sqrt{y_1}}{\sqrt{y_1} + \sqrt{y_2}} \quad \text{.......... (v)}$$

Substituting the value of ℓ_1 (v) in Eq. (ii)

$$H = \frac{w\ell^2}{2\left(\sqrt{y_1} + \sqrt{y_2}\right)^2}$$

7.7 Suspension Bridge with Three-Hinged Stiffening Girder

When suspension a bridge is used in large spans with the provision of pavement for rolling loads, these bridges are stiffened with three hinged or two hinged stiffening girder as shown in Fig. (7.25). The purpose of the providing these stiffening girders is to reduce the sag under the rolling load.



Suspension bridge with three hinged stiffening girder

Fig. 7.25

As there will be no distortion in the pavement due to the stiffening girder, the cable will retain its parabolic shape under the application of load. When the bridge is loaded with uniformly distributed load, the cable directly takes whole the load and there will be no effect on the stiffening girder. But the girder transfers unsymmetrical loads as a uniformly distributed load through

suspenders. So, the stiffening girder subjected to bending moments and shear forces.

7.8 BM and SF in Three-Hinged Stiffening Girder

Let us consider a suspension bridge with a three hinged girder loaded with point loads as shown W as shown in Fig (7.26). The load W is transferred to the cable as equivalent uniformly distributed (w_e) through suspenders. w_e is determined by using $\Sigma M_c = 0$ (about the hinge). After this, the beam can be analysed for the given loading as illustrated below,

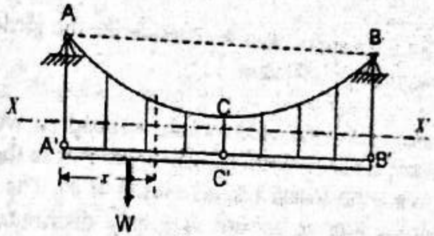


Fig. 7.26

Let us consider a suspension bridge where a moving load W acts at distance a from A

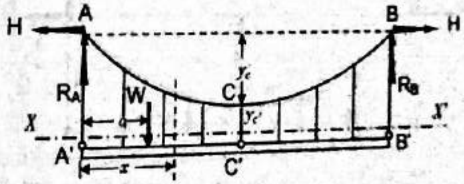


Fig. 4.27

Taking moment about A, $R_B = \frac{Wa}{\ell}$

and $R_A = \frac{W(\ell - a)}{\ell}$

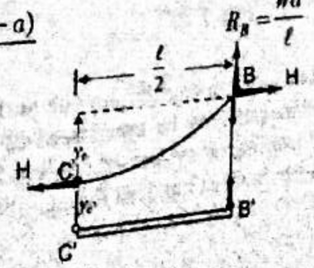


Fig. 7.28

Consider the equilibrium of right portion of the bridge and taking moment about C, we have

$$M_c' = 0 \;\Rightarrow\; \frac{Wa}{\ell}\times\frac{\ell}{2} + Hy_c' - H(y_c + y_c') = 0$$

or, $\dfrac{Wa}{2} + Hy_c' - Hy_c - Hy_c' = 0, \qquad \therefore H = \dfrac{Wa}{2y_c}$

$$M_x = R_A x - W(x-a) - Hy \qquad \text{.......(7.11)}$$
$$= M\,beam - Hy$$
$[\because R_A x - W(a-x) =$ Simply supported beam moment$]$

We can also arrive at the above result by considering the girder system cut by a section as shown in Fig. (7.29) also.

Once the structure is cut at the middle of the suspenders, we will be able to see two independent structural portions. The upper part is the cable structure loaded with uniformly distributed load (as a result of W). The lower part is the beam system loaded with an upward uniformly distributed load and the external load W as shown below.

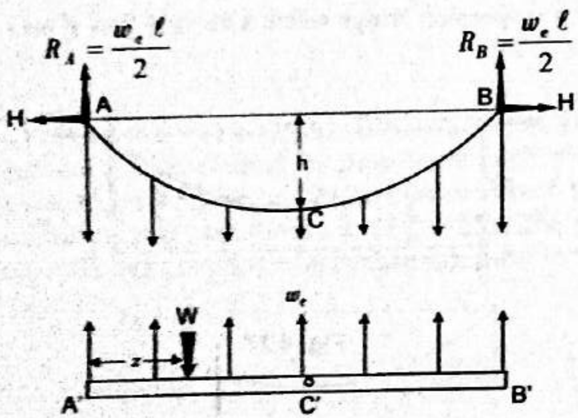
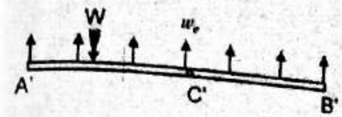


Fig. 7.29

Obviously, the lower system is not in equilibrium due to the eccentric system of loading. The lower beam can now be considered as comprised of two beams with two types of loading: W system of loading and w_e (udl). Let us separate them and add the reactions as required for equilibrium. .



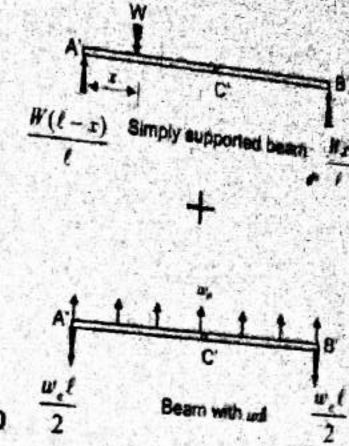


Fig. 7.30

Moment at x is

$$M_x = M_{beam} - \left(\frac{w_e\ell}{2}x - w_e\frac{x^2}{2}\right)$$
$$= M_{beam} - \frac{w_e x}{2}(\ell - x)$$
$$= M_{beam} - Hy$$

$\therefore \dfrac{w_e x}{2}(\ell - x) = Hy$ as shown below

We know, $H = \dfrac{w_e\ell^2}{8y_c}$ and $y = \dfrac{4y_c x}{\ell^2}(\ell - x)$

$$Hy = \frac{w_e\ell^2}{8y_c}\cdot\frac{4y_c x}{\ell^2}(\ell - x)$$
$$= \frac{w_e x}{2}.(\ell - x)$$

Thus the *BM* diagram for the girder will be the superimposition of *BM* diagram due to beam moment and due to $H.y_c$. Value of H for a particular loading is constant and hence for Hy_c the diagram will be a parabola. The diagram for the Hy_c is obtained by taking the parabolic shape of cable with every ordinate multiplied by H which is as shown below

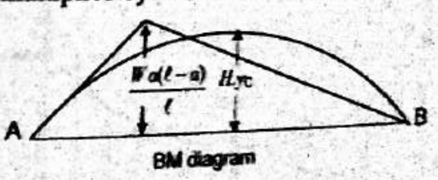


Fig. 7.31

Similarly for shear force at x,

$$F_x = F_{beam} - \frac{w_e\ell}{2} + w_e x$$
$$= F_{beam} - \frac{w_e}{2}(\ell - 2x)$$
$$= F_{beam} - H\tan\theta$$

($\because \frac{w_e}{2}(\ell - 2x) = H \tan\theta$ which is proved below.)

$$y = \frac{4y_c x}{\ell^2}(\ell - x)$$

$$\frac{dy}{dx} = \tan\theta = \frac{4y_c}{\ell^2}(\ell - 2x) \text{ and } H = \frac{w_e \ell^2}{8h}$$

$$\therefore H\tan\theta = \frac{w_e \ell^2}{8h} \cdot \frac{4y_c}{\ell^2}(\ell - 2x)$$

$$= \frac{w_e}{2}(\ell - 2x)$$

Example # 7.15 Calculate the load in the main cable of suspension bridge due to a point load of 4 *kN* acting at a distance of 25 *m* from the left support on the three- hinged stiffening girder of the bridge. The span of the bridge is 100 *m* with the intermediate hinge at the mid-span.

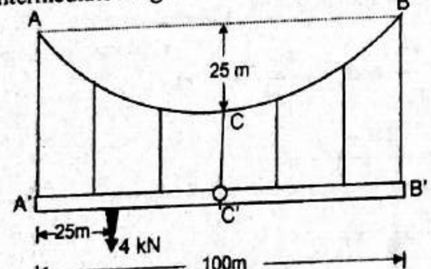


Fig. 7.32

Sol^n. Cutting the bridge along the mid of the suspenders,

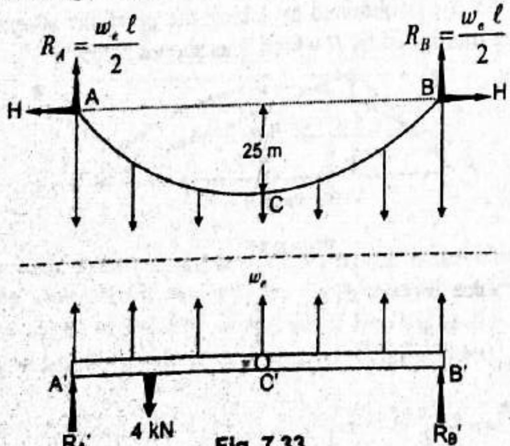


Fig. 7.33

Let w_e be the load on the cable due to the point load of 4 *kN*, then,

$$\Sigma M_c = 0, \quad \text{or, } R'_B \times 50 + w_e \frac{50^2}{2} = 0$$

or, $50R'_B + 1250 w_e = 0$ .... (i)

$$\Sigma M_A = 0, \quad \text{or, } R'_B \times 100 + w_e \times \frac{100^2}{2} - 4 \times 25 = 0$$

or, $100R'_B + 5000 w_e = 100$ .... (ii)

or, $50R'_B + 125 w_e = 0$ .... (iii)

Solving equation (i) and (ii), we get

$$2500 w_e = 100$$

$$\therefore \quad w_e = \frac{100}{2500}$$

$$= \frac{1}{25} \; kN/m$$ Ans.

Example # 7.16 A three hinged stiffening girder of a suspension bridge of span 100 *m* is subjected to two point loads of 200 *kN* and 300 *kN* at the distances of 25 *m* and 50 *m* from the left end. Find the shear force and bending moment for the girder at a distance 30 *m* from the left end. The supporting cable has a central dip of 10 *m*. Find also the maximum tension and its slope in the cable.

Sol^n. Let *A' C' B'* be the beam girder where two point loads 200 *kN* and 300 *kN* are applied. The suspenders of the bridge extents an upward uniformly distributed load of w_e per unit horizontal length as shown. Now we can see the girder which is supported at *A'* and *B'* on by upward uniformly distributed load and downward point loads as shown in Fig. (7.34).

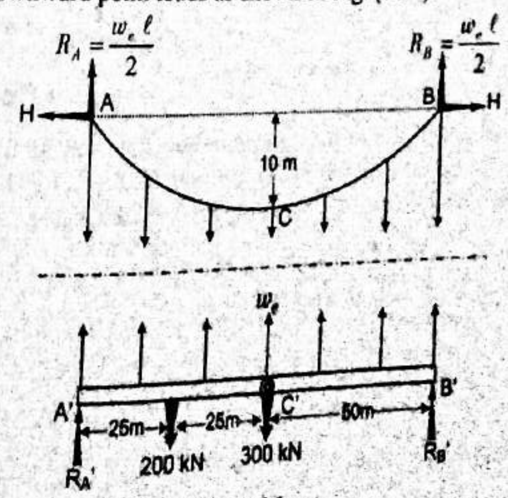


Fig. 7.34

Let R'_A and R'_B are the required reactions to be produced to keep the girder in vertical equilibrium due to the system of loading.
Now, take moment about A, $\sum M_A = 0$,

$$R'_B \times 100 + w_e \times \frac{100^2}{2} = 300 \times 50 + 200 \times 25$$

or, $100\ R_B + 5000\ w_e = 20000$ .... (i)

$\sum M_c = 0$,

$$R'_B \times 50 + w_e \frac{50^2}{2} = 0$$

or, $50\ R_B + 1250\ w_e = 0$ .... (ii)

$100\ R_B + 5000\ w_e = 20000$ .... (i)

Solving (i) and (ii), we get, $w_e = 8\ kN/m$

Substituting the value of w_e in equation (i), we get

$$R_B = \frac{20000 - 40000}{100} = -200\ kN\ (\downarrow)$$

Again, we have $\sum Fy = 0$,

$$R'_A + R'_B + w_e \times 100 - 200 - 300 = 0$$

or, $R_A = 500 - 8 \times 100 + 200 = 100\ kN\ (\downarrow)$

Now, shear force at 30 m from left can be determined from the section shown below.

Unbalanced vertical force $= -100 - 200 + 8 \times 30$

$= -300 + 240 = -60\ kN$

$\therefore$ S.F. at the section $= +60\ (\downarrow)$ Ans.

$$B.M. = 100 \times 30 + 200 \times 5 - 8 \times \frac{30^2}{2}$$

$= 400\ kN-m$ (hogging) Ans.

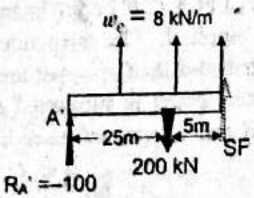


Fig. 7.35

The horizontal reaction H of the point A in the cable can be found out from the free body diagram of the upper portion in Fig. (7.34). Since the uniformly distributed load of w_e is applied in the cable, from previous formula,

$$H = \frac{w_e l^2}{8 y_c} = \frac{8 \times 100^2}{8 \times 10} = 1000\ kN$$

and $R_A = \frac{w_e l}{2} = \frac{8 \times 100}{2} = 400\ kN$

Now, the maximum tension

$$T_{max} = \sqrt{R^2_A + H^2}$$

$= \sqrt{400^2 + 1000^2}$

$= 1077.03\ kN$ Ans.

Slope to the horizontal is given by

$$T_{max} \cos\theta = H$$

$$\theta = \cos^{-1} \frac{1000}{1077.03} = 21.8^0 \text{ Ans.}$$

Example # 7.17 A three hinged stiffened girder of a suspension bridge of span 400 m is subjected to three point loads of 300, 500 and 400 kN acting at 70 m, 160 m and 300 m respectively from the left hand support. The supporting cable has a central dip of 40 m. Draw the $B.M.$ diagram for the girder.

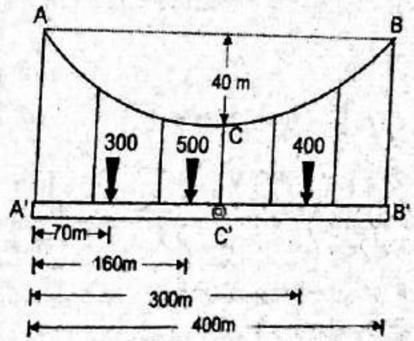


Fig. 7.36

Sol^n. Let us cut the structure at the center of the suspenders and draw free body diagram.

$\sum M_{A'} = 0$

or, $R_B \times 400 + w_e \frac{400^2}{2} - 400 \times 300 - 500 \times 160 - 300 \times 70 = 0$

or, $400\ R'_B + 80000\ w_e = 221000$ .... (i)

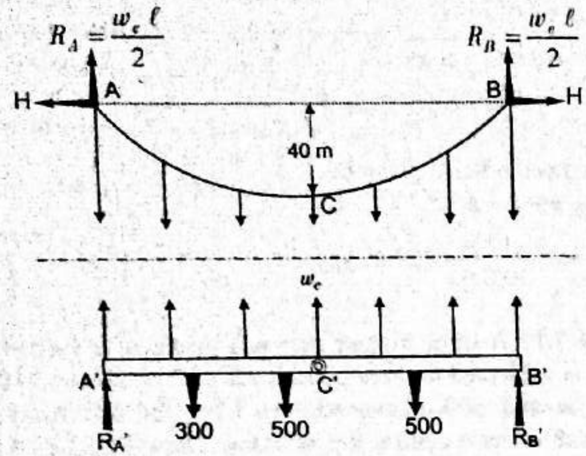


Fig. 7.37

Similarly, $\sum M_c = 0$

or, $200 \times R_B + w_e \times \frac{200^2}{2} - 400 \times 100 = 0$

or, $200\ R_B' + 20000\ w_e = 40000$ .... (ii)

Solving (i) and (ii) we get,

$w_e = 3.525\ kN/m$

Substituting the value of w_e in Eq. (i) gives .

$$R_{B'} = \frac{221000 - 80000 \times 3.525}{400} = -152.5kN$$

$\sum Fy = 0$ gives,

$R_A' + R_B' + w_e \times 400 = 1200$ .... (iii)

Substituting the value of R_B and w_e in Eq. (ii) we get,

$R_A = -57.5\ kN$

Now, moment at section of 70 m distance from A' is,

$$M_{70} = R_A' \times 70 + w_e \frac{\ell^2}{2}$$

$$M_{70} = -57.5 \times 70 + 3.525 \times \frac{70^2}{2} = 4611.25\ kN-m$$

Similarly,

$$M_{160} = -57.5 \times 160 + 3.525 \times \frac{160^2}{2} - 300 \times 90 = 8920\ kN-m$$

$$M_{300} = -57.5 \times 300 + 3.525 \times \frac{300^2}{2} - 300 \times 230 - 500 \times 140$$

$$= 2375\ kN-m$$

$$H = \frac{w_e \ell^2}{8 y_c} = \frac{3.525 \times 400^2}{8 \times 40} = 1762.5$$

$$y = \frac{4 y_c x}{\ell^2}(\ell - x)$$

The diagram for $H.y$ is parabolic and it is shown in Fig. (7.38).

The values of product of H and y i.e. the points on parabola may also be determined if wished to.

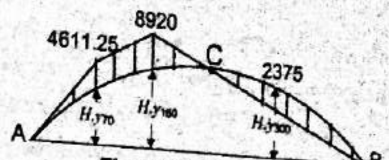


Fig. 7.38

Example # 7.18 A suspension bridge of 100 m span has a three-hinged stiffening girder supported by cables having a central dip of 10 m as shown in Fig. (7.39). The left half span of the bridge is loaded with a uniformly distributed load of 30 kN/m. Determine the reaction and draw the bending moment diagram.

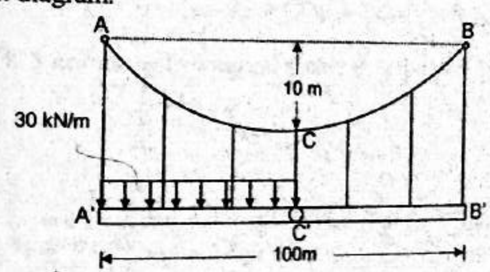


Fig. 7.39

Solⁿ. Let us cut the structure at the center of the suspenders and draw the force diagram.

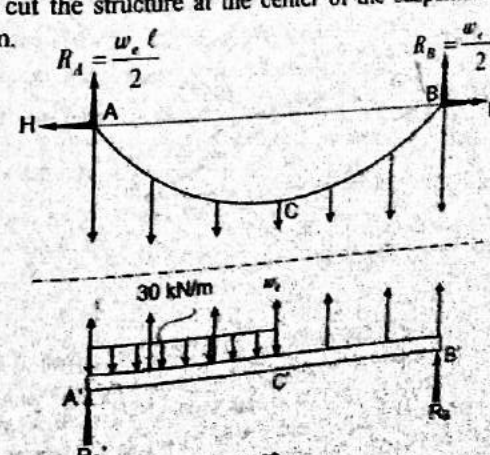


Fig. 7.40

Iⁿ. Taking moment about B'

$$R_A = \frac{1}{100} \times \left(30 \times 50 \times 75 - w_e \times \frac{100^2}{2}\right) = 1125 - 50w_e \quad \text{.......... (i)}$$

Taking moment about C', $R_B \times 50 - 30 \times \frac{50^2}{2} + w_e \times \frac{50^2}{2} = 0$

$$\text{or, } R_B = 750 - 25w_e \quad \text{.......... (ii)}$$

Solving Equation (i) and (ii), $w_e = 15\ kN/m$ and

$R_A = 1125 - 50 \times 15 = 375\ kN$ (upward)

$R_B = -25 \times 15 = -375\ kN$ (downward)

Let the maximum bending moment occur in $A'C'$ at a distance x from A.

The bending moment, $\quad M_x = 375x - (30 - 15) \times \frac{x^2}{2}$

For maximum value of M_x $\quad \frac{dM_x}{dx} = 375 - 15x = 0$

or, $\quad x = 25\ m$

$$M_{max}(+) = 375 \times 25 - 15 \times \frac{25^2}{2} = 4687.5\ kN-m$$

Let the maximum negative bending moment occurs in the portion $C'B'$ at a distance x from B'

The bending moment, $\quad M_x = 375x - 15 \times \frac{x^2}{2}$

For maximum value of M_x $\quad \frac{dM_x}{dx} = 375 - 15x = 0 \quad$ or, $x = 25\ m$

$$M_{max}(-) = 375 \times 25 - 15 \times \frac{25^2}{2} = 4687.5\ kN-m$$

The BMD is shown below.

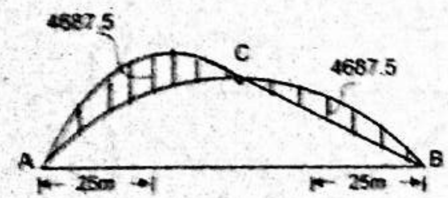


Fig. 7.41

Example # 7.19 The stiffening girder of a suspension bridge of span 120 m has hinges at the ends and at mid-span. The cable is suspended between two points separated horizontally by 120 m and vertically by 10 m, as shown in Fig. (7.42). The maximum dip of the cables is 10 m low the lower suspension point. Draw shear force and bending moment diagram for the girder due to a concentrated load of 180 kN acting at the central hinge. Also calculate the maximum tension in the cable.

Sol^n. Taking moment about B'

$$R_A = \frac{1}{120} \times \left(180 \times 60 - w_e \times \frac{120^2}{2}\right) = 90 - 60w_e \quad \text{.......... (i)}$$

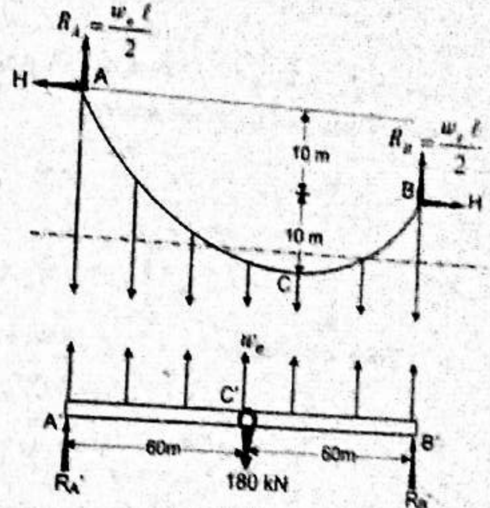


Fig. 7.42

Taking moment about C', $\quad R_B \times 60 + w_e \times \frac{60^2}{2} = 0$

or, $R'_A = -30w_e$

Solving Equation. (i) and (ii), $w_e = 3\ kN/m$ and

$R_A = 90 - 60 \times 3 = -90\ kN$ (downward)

$R_B = -90\ kN$ (downward)

Let the maximum bending moment occur in portion $A'C'$ at x from A'.

The bending moment $\quad M_x = -90x + 3 \times \frac{x^2}{2}$

For maximum value of M_x $\quad \frac{dM_x}{dx} = -90 + 3x = 0 \quad \therefore x = 30\ m$

$$M_{max} = -90 \times 30 + 3 \times \frac{30^2}{2} = -1350\ kN-m$$

Shear force and bending moment diagrams for the girder are shown in Fig. (7.43). Horizontal distance of the lowest point of the cable from A

$$= \frac{120 \times \sqrt{20}}{\left(\sqrt{10} + \sqrt{20}\right)} = 70.294$$

Vertical reaction at A.

$$R_A = 3 \times 70.294 = 210.88\ kN$$

Horizontal component of cable tension

$$H = \frac{3 \times 120^2}{2\left(\sqrt{10} + \sqrt{20}\right)^2} = 370.6 kN$$

Maximum tension in the cable $= \sqrt{\left(210.88^2 + 370.6^2\right)} = 426.4\ kN$ Ans.

Shear force and bending moment diagrams are shown below.

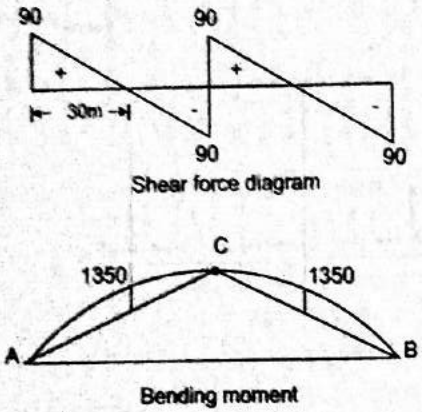


Fig. 7.43

Example # 7.20 A suspension cable with 100 m span and 8 m dip is stiffened by a three-hinged girder. The dead load of the girder and deck is 7.5 kN/m. Find $S.F.$ and $B.M.$ in the girder at a section 20 m from left hand hinge when a concentrated load of 150 kN is placed at 16 m from the left end. Find the maximum tension in the cable.

Sol^n.

Given

Span $\ell = 100\ m$

Central dip $y_c = 8\ m$

Dead load $(w) = 7.5\ kN/m$

Concentrated load $(W) = 150\ kN$

Now, due to $D.L.$ $H = \frac{w\ell^2}{8y_c} = \frac{7.5 \times 100^2}{8 \times 8} = 1171.875\ kN$

$$R_A = \frac{w\ell}{2} = \frac{7.5 \times 100}{2} = 375\ kN$$

Due to concentrated load $W = 100\ kN$, the value of H can be obtained by knowing the value of w on the cable.

Taking moment about the section 20 m from left support.

$$M_x = \frac{w_e \ell}{2} \times 20 - w_e.\frac{(20)^2}{2} - H.y = 0$$

$$\text{or, } H.y = \frac{w_e \ell}{2} \times 20 - w_e.\frac{20^2}{2}$$

$$= w_e.\frac{100}{2} \times 20 - w_e \times \frac{400}{2}$$

$$= w_e.1000 - w_e.200$$

$$\therefore H.y = 800\ w_e \quad \text{.......... (i)}$$

As $x = 20\ m$,

$$y = \frac{4y_c x}{\ell^2}(\ell - x)$$

$$= \frac{4 \times 8 \times 20}{100^2}(100 - 20)$$

$$= 5.12 m$$

Again, we know $w_e = \frac{4WZ}{\ell^2} = \frac{4 \times 150 \times 16}{100^2} = 0.96\ kN/m$

Now from equation (i)

$$H = \frac{800 \times 0.96}{5.12} = 150\ kN$$

Again, vertical reaction due to 150 kN load $(R_A) = \frac{0.96 \times 100}{2} = 48\ kN$

Total horizontal pull due to dead load and concentrated load

$$= 1171.88 + 150$$

$$= 1321.88\ kN$$

Total vertical reaction $R_A = 375 + 48$

$$= 423\ kN$$

Tension force on cable $T = \sqrt{R_A^2 + H^2}$

$$= \sqrt{423^2 + 1321.88^2}$$

or, $T = 1387.91\ kN$

Now, consider the girder.

Uniformly distributed dead load does not cause any shear or moment on the girder. S.F. at section X,

$$V_x = -[V_x + H\tan\theta x]$$

$$= -\left[150\times\frac{16}{100} + 150\times\tan\theta_x\right]$$

$$= -\left[150\times\frac{16}{100} + 150\times 0.192\right]$$

$$= -52.8\ kN$$

$$y = \frac{4y_c}{\ell^2}(\ell - x)$$

$$\frac{dy}{dx} = \frac{4y_c x}{\ell^2}(\ell - 2x)$$

$$\tan\theta_x = \frac{4\times 8}{100^2}(100 - 2\times 20)$$

$$= 0.192$$

B.M. at section X, using equation.

$$M_x = M_c x - Hy$$

Where y is the dip of the cable above the section.

$$\therefore M_x = 48\times 20 - 150\times 5.12$$

$$= 192\ kN\text{-}m$$

7.9 *ILD* THREE HINGED STIFFENING GIRDERS

(i) *ILD* for Horizontal thrust

Let the unit load is at a distance y from A' in the three hinged girder arrangement shown in the Fig. (7.44). Taking moment about C,

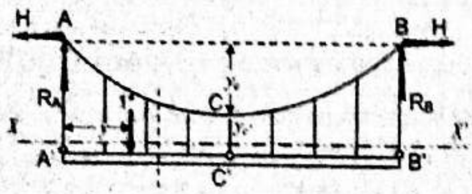


Fig. 7.44

$$H.y_c - R_A\times\frac{\ell}{2} + w_c\frac{\left(\frac{\ell}{2}\right)^2}{2} = 0$$

$$\text{or } H = \frac{w_c\ell^2}{8y_c} \quad \text{.......................................(7.12)}$$

The expression for w_e can be obtained by taking moment about C'. Considering forces to the right of C' only.

$$R_B\frac{\ell}{2} - w_e\frac{\left(\frac{\ell}{2}\right)^2}{2} = 0 \quad \left(\text{But } R_B = \frac{x}{\ell}\right)$$

$$\therefore w_e = \frac{4x}{\ell^2}$$

Substituting the value of w_e in Eq. (7.12), we get

$$H = \frac{4x}{\ell^2}\cdot\frac{\ell^2}{8y_c} = \frac{x}{2y_c}$$

When $x = 0$, $H = 0$

When $x = \frac{\ell}{2}$, $H = \frac{\ell}{4y_c}$

ILD is drawn in Fig. (7.45)

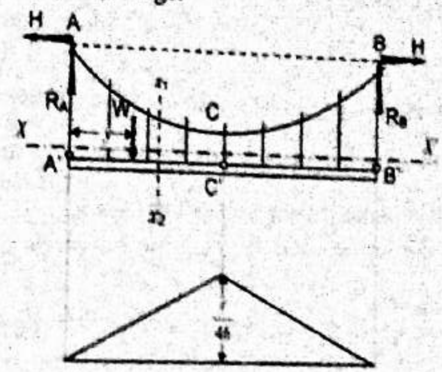


Fig. 7.45

ii) Influence lines for Bending moment

As explained in section -7.8, moment at a section of girder is given by.

$$M_x = M_{Beam} - Hy$$

The relation is valid for a load of unit magnitude also. *ILD* of M_{Beam} for the section x is known to us. Its ordinate is given by

$$DD_1 = \frac{x(\ell - x)}{\ell}$$

The ordinate of *ILD* for Hy is given by

$$CC_1 = \frac{\ell}{4y_c}\cdot\frac{4y_c x(\ell - x)}{\ell^2} \quad \left[\because y = \frac{4y_c x(\ell - x)}{\ell^2}\right]$$

$$= \frac{x(\ell - x)}{\ell}$$

ILD is thus superimposition of M_{Beam} and Hy as shown below.

iii) *ILD for shear force*

As explained in section 7.8, shear force at any section of the girder is given by

$$F_x = F_{Beam} - H\tan\theta$$

$$= F_{Beam} - \frac{w_e}{2}(\ell - 2x)$$

$$= F_{Beam} - \frac{4x}{\ell^2}\left(\frac{\ell}{2} - x\right)$$

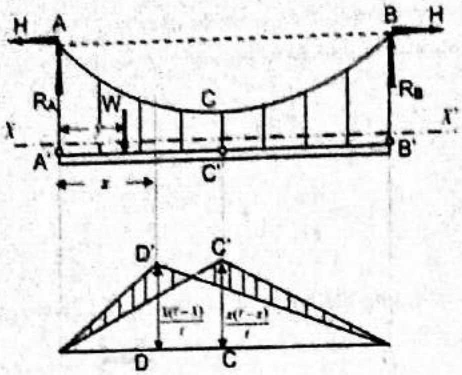


Fig. 7.46

The relation is valid for a load of unit magnitude also. *ILD* for the section x for beam is known to us. Its ordinates are $\frac{x}{\ell}$ and $\frac{1}{\ell}(\ell - x)$ in positive and negative sides or the ordinates 1 at A and B.

For the second term $\frac{4x}{\ell^2}\left(\frac{\ell}{2} - x\right)$, we can obtain a diagram by using various values of x.

When $x = 0$, $\quad SF = 0$

When $x = \frac{\ell}{2}$ $\quad SF = \frac{\ell - 2x}{\ell}$

The diagrams for the two terms are superimposed and the resulting diagram is shown below.

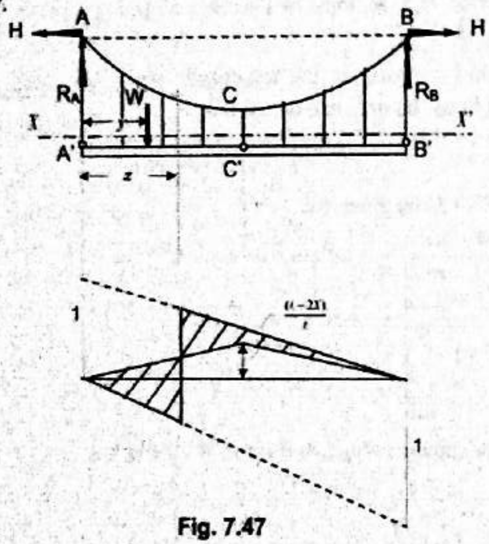


Fig. 7.47

Example # 7.21 A cable of suspension bridge has a span of 50 m and a dip of 6 m. The cable is stiffened by a girder hinged at the ends and at the mid–span.

There is a uniform dead load of 12 kN/m over the whole span and a live load 20 kN/m spread over 15 m length. Determine the maximum cable tension when the tail of live load is on the central hinge. [T.U. 2059 Jeshtha]

Sol^n.

Given,

Span $\quad \ell = 50\ m$

Central dip $\quad y_1 = 6m$

Uniform dead load $= 12kN/m$ (over whole span)

Live load $= 20kN/m$ (Over 15 m length)

<u>Horizontal reactions H_1 and H_2</u>

Horizontal tension on the cable due to $D.L.$ $H_1 = \frac{w\ell^2}{8y_c} = \frac{12\times50^2}{8\times6} = 625\ kN$

Horizontal tension on the cable due to $L.L.$, $H_2 = \frac{M_c}{y_c}$

M_C is the moment in a simply supported beam at the centre C due to the given live load position.

$$M_C = R_A \times 25$$

Now, $\quad R_A = \frac{Wb}{\ell} = \frac{20\times15\times17.5}{50} = 105\ kN$

$\therefore \quad M_C = 105\times25 = 2625\ kN\text{-}m$

Substituting the values,

$$H_2 = \frac{2625}{6} = 437.5\ kN$$

<u>Vertical reactions R_{A1} and R_{A2}</u>

Vertical Reaction R_A due to $D.L.$ $R_{A1} = \frac{w_1\ell}{2} = \frac{12\times50}{2} = 300\ kN$

If equivalent *udl* on the cable be w_e due to live load.

$$\therefore \frac{w_e\ell^2}{8y_c} = H_2$$

$$\therefore w_e = \frac{H_2\times8y_c}{\ell^2} = \frac{8\times437.5\times6}{50^2}$$

$$= 8.4\ kN/m$$

So, vertical reaction due to $L.L.$

$$R_{A2} = \frac{w_e\ell}{2} = \frac{8.4\times50}{2} = 210\ kN$$

$\therefore$ Total vertical reaction $R_A = R_{A1} + R_{A2}$

$=(300+210)\ kN$

$=510\ kN$

Total horizontal reaction $H=H_1+H_2$

$=(625+437.5)\ kN$

$=1062.5\ kN$

Maximum tension of the cable

$$T_{max}=\sqrt{R_A^2+H^2}$$

$$=\sqrt{510^2+1062.5^2}$$

$$=1178.56\ kN$$

Example # 7.22 The towers of a 200 m span suspension bridge are of unequal heights. One tower is 24 m and the other is 8 m above the lowest point of the cable, which is immediately above the inner hinge of a three–hinged stiffening girder. Find the maximum tension in the cable due to point load W rolling over the bridge.

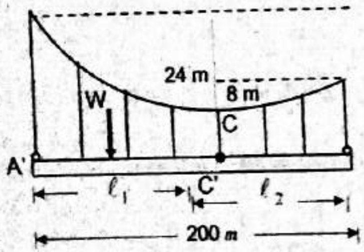


Fig. 7.48

Sol^n.

Let us first locate the position of the lowest point C of the cable. Let it be at the distances ℓ_1 and ℓ_2 from end A and B respectively.

Then,

$$\frac{\ell_1}{\ell_2}=\left(\frac{24}{8}\right)^{\frac{1}{2}} \qquad \therefore \frac{\ell_1}{\ell_2}=\left(\frac{y_c+d}{y_c}\right)^{\frac{1}{2}}$$

or, $$\frac{\ell_1}{\ell_2}=1.73$$

$$\therefore \quad \ell_1=1.73\ell_2 \quad \text{.......... (i)}$$

We know that

$$\ell_1+\ell_2=200$$

or, $$\ell_1+1.73\ell_2=200$$

or, $$\ell_2=\frac{200}{2.73}=73.2\ m \quad \text{.......... (ii)}$$

Substituting this value in Eq. (i) above, we get,

$$\ell_1=200-\ell_2$$

$$=(200-73.2)\ m$$

$$=126.8\ m$$

Taking moment about C of all forces to the right of C

$$M_c=\mu_x-H.y-H.d\left(\frac{\ell_2}{\ell}\right)=0$$

The last term in the above equation is due to the towers being at different levels.

$$\therefore M_c=\frac{W(\ell_2.\ell_1)}{\ell}-H.y_c-H.d\left(\frac{\ell_2}{\ell}\right)=0$$

or, $$\frac{W\times 73.2\times 126.8}{200}-H\times 8-H\times 16\times\frac{73.2}{200}=0$$

or, $46.41W-13.856\ H=0$

or, $$H=\frac{46.41W}{13.856}$$

$$=3.35W$$

Again, we know

$$H=\frac{w\ell_2^2}{2y_c}$$

$$\therefore w=\frac{H.2y_c}{\ell_2^2}$$

$$=\frac{2\times 3.35W\times 8}{(73.2)^2}$$

$$=0.01W$$

Vertical reaction at end A of the cable is obtained by taking moment about C.

$$R_A\times 126.8-H\times 24-0.01\times\frac{126.8^2}{2}W=0$$

or, $$R_A=\frac{\left[3.35W\times 24+0.01\times\frac{126.8^2W}{2}\right]}{126.8}$$

$$=1.27W$$

Maximum Tension

$$T_{max} = \sqrt{R_A^2 + H^2}$$
$$= \sqrt{(1.268W)^2 + (3.35W)^2}$$
$$= 3.58W$$

Example # 7.23 A three hinged symmetrical stiffening girder of a suspension bridge of 100 *m* span is subjected to the two points loads of 100 *kN* each placed at 20 *m* and 40 *m* respectively from the left hand hinge. Determine the maximum tension in the cable which has the central dip of 10 *m*. Determine the length of the cable and also *B.M.* and *S.F.* in the girder at the section 30 *m* from left end. [T.U.2059 Shrawan]

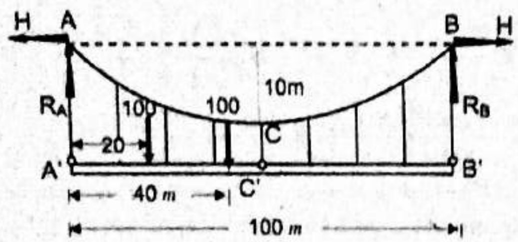


Fig. 7.49

Sol^n.

Now, to find out the support reaction.

$$R_B \times 100 = 100 \times 20 + 100 \times 40$$

$\therefore$ $R_B = 60\ kN$

since, $R_A + R_B = 100 + 100\ kN$

$\therefore$ $R_A = (200 - 60)\ kN$
$= 140\ kN$

To get horizontal thrust

Take moment about C,

$$M_C = R_B \times 50$$
$$= 160 \times 50$$
$$= 3000\ kN\text{-}m$$

Using relation

$$H = \frac{M_C}{y_c}$$
$$= \frac{3000}{10}$$
$$= 300\ kN$$

$\therefore$ Maximum Tension

$$T_{max} = \sqrt{R_A^2 + H^2}$$
$$= \sqrt{140^2 + 300^2}$$
$$= 331.06\ kN$$

Again length of the cable

$$= \ell + \frac{8y_c^2}{3\ell} = 100 + \frac{8 \times 10^2}{3 \times 100} = 102.67\ m$$

<u>*S.F.* and *B.M.* moment</u>

To find out shear force at 30 *m*,

$$= F_x + H \tan\theta$$

Now, $y = \frac{4y_c x}{\ell^2}(\ell - x)$

Differentiating, $\frac{dy}{dx} = \frac{4y_c}{\ell^2}(\ell - 2x)$

or, $x = 30$,

$$\frac{dy}{dx} = \tan\theta = \frac{4 \times 10 \times (100 - 2 \times 30)}{100^2} = 0.16$$

$$\therefore F = F_x + H \tan\theta$$
$$= (140 - 100) + 300 \times 0.16$$
$$= 40 + 48$$
$$= 88\ kN$$

B.M. at the 30 *m* from left hand support

Now, $y = \frac{4y_c x}{\ell^2}(\ell - x)$

or, $x = 30\ m$, $y_c = 10\ m$ and $\ell = 100\ m$

$$y = \frac{4 \times 10 \times 30}{100^2}(100 - 30)$$
$$= 8.4\ m$$

B.M. at 30 *m* due to external loading

$$= R_A \times 30 - 100 \times 10$$
$$= 140 \times 30 - 100 \times 10$$
$$= 3200\ kN\text{-}m$$

B.M. at 30 *m* due to horizontal thrust

$$= 300 \times 8.4$$
$$= 2520\ kN\text{-}m$$

$\because$ *B.M.* at the 30 *m*

$$= (3200 - 2520)\ kN\text{-}m$$
$$= 680\ kN\text{-}m$$ **Ans**

7.10 EXERCISE

Ex. 1 A suspension cable is supported at two points $20\ m$ apart. The left support is $2\ m$ above the right support. The cable is loaded with uniformly distributed load of $10\ kN/m$ throughout the span. The maximum dip in the cable from left support level is $4\ m$. Find the maximum tension in the cable

(Ans: 208 kN)

Ex. 2 A suspension cable of uniform section, is hung in the form of a parabola. Find the maximum horizontal span, if the central dip is $1/12^{th}$ of the span and the stress in cable is not to exceed $1200\ kg/cm^2$. Take density of steel $= 7800\ kg/m^3$

(Ans: 955.5 cm)

Ex. 3 A suspension bridge with three hinged stiffening girder has span of $100\ m$ and central dip of $10\ m$. The self-weight of bridge carried by one set of cables is 15 kN. The bridge is to be designed to carry a live load of 30 kN/m to be equally divided between set of two cables. The working stress is 15 kN/cm^2 for cables and 120 kN/cm^2 for girder. Find (a) cross sectional area of one set of suspension cables and (b) necessary section modulus of the stiffening girder.

(Ans: $269.24\ cm^2$, $3{,}538\ cm^3$)

Ex. 4 A cable of uniform thickness hangs between two points $120\ m$ apart, with one end $3\ m$ above the other. The cable is loaded with uniformly distributed load of $100\ kN/m$ and the sag of the cable, measured from the higher end, is $5\ m$. Find the horizontal thrust and maximum tension in the cable.

(Ans: 54025 kN, 54500 kN)

Ex. 5 A cable is suspended between two points A and B, $100\ m$ apart horizontally, end B bending $5\ m$ lower than the end A. It supports a uniform load of w per unit horizontal length. Determine
a) The position of lowest point, if cable has a sag of 4,5 m below the support B.
b) The length of the cable.
c) The horizontal tension and maximum tension at the two ends of the cable.

(40.77 m from lower end, $S = 101.35$ m,
$H = 184.6\ w_e$, $T_A = 193.9\ w_e$, $T_B = 189\ w_e$)

8

SPACE TRUSSES

8.1 INTRODUCTION

A space truss is one whose members do not all lie in the same plane. When several members are joined together at their extremities to form a three dimensional configuration, we obtain a space truss.

An elementary two-dimensional truss consists of three members joined at their extremities to form the sides of a triangle. By adding two members at a time to this configuration and connecting them at a new joint, it is possible to obtain a larger two dimensional truss. Similarly, the most elementary rigid space truss consists of 6 members joined at their extremities to form the edges of a tetrahedron $ABCD$. By adding three members at a time to this basic configuration such as AE, BE and CE, attaching them to three existing joints and connecting them at a new joint, we can obtain a larger and rigid structure which is called a simple space truss.

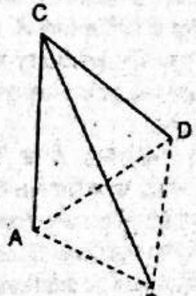


Fig. 8.1

The relation between number of joints (j) and members (m) is given by

$$3j = m + r.$$

Where r = numbers of independent reaction components. When this relation is used, supports are not counted as joints. There are generally three types of supports used in space trusses as shown in figure below. It can also be noted that if $3j > m + r$ the structure is statically unstable and of $3j < m + r$ the structure is statically indeterminate.

8.2 Types of Supports

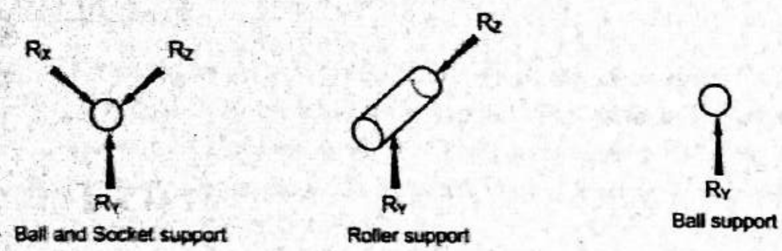


Fig. 8.2

As clear from the figure, the ball and socket support prevents the movement of the support in each of the three directions while roller and ball supports prevent it in two and one directions respectively.

8.3 Analysis of Space Trusses

The equilibrium of an entire space truss or its section is given by the following six equations

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum F_z = 0$$

and $$\sum M_x = 0 \qquad \sum M_y = 0 \qquad \sum M_z = 0$$

If a space truss is supported only on three points, then applying the above sex equations can provide all six unknown reaction forces. However if there are more than three support points, it is usually necessary to determine some or all of the bar forces before the reactions can be evaluated.

The bar forces are determined by applying three equilibrium equation $\sum F_x = 0$, $\sum F_y = 0$ and $\sum F_z = 0$ at each joint. In a simple truss, the analysis can be started at a joint where only three members are connected. The direction cosines, which are necessary to find the resultant force in the members, are obtained from the coordinates of the joints. The relations are given below.

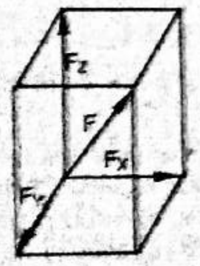


Fig. 8.3

If ℓ_x, ℓ_y and ℓ_z are the projections on x, y and z axis, then the length of the member is given by

$$\ell = \sqrt{\ell x^2 + \ell y^2 + \ell z^2}$$

Similarly, if F_x, F_y and F_z are the components of the force F, then the relation between the force and their components can be expressed as

$$F_x = F\frac{\ell_x}{\ell},\ F_y = F\frac{\ell_y}{\ell} \text{ and } F_z = \frac{\ell_z}{\ell}$$

where $\frac{\ell_x}{\ell}$ are also called direction cosines.

In analyzing space trusses, the following two theorems may also be used which help to save considerable computational efforts.

Theorem 1

At a joint having only three members not in are plane and where external forces are zero, the forces in all members meeting at that joint must be zero.

Theorem 2

At a joint where more members meet but where all members except one lie in one plane, then the force in this member will be zero if there is no external load at that joint. If there is load, then the perpendicular component of force in that member (out of plane) will be equal to the normal component of the load.

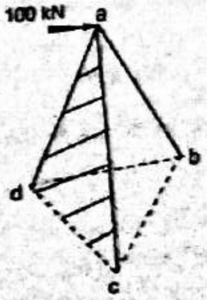


Fig. 8.4

Considering joint, the members ad and ac lie in one plane, ab is the out of place member. Therefore $F_{ab} = 0$

Example # 8.1 Determine the reactions and bar forces for the space truss given below.

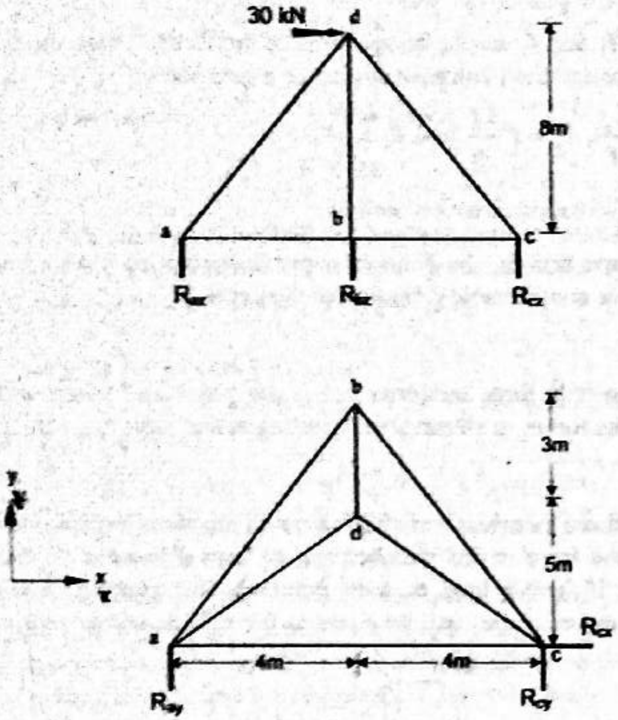


Fig. 8.5

Taking moment about the line *ac* which is the axis passing through two supports *a* and *c*.

$\sum M_z = 0$ about ac, i.e. $R_{bz} = 0$

$\sum M_y = 0$ about line of action of R_{cy}

$R_{az} \times 8 + 15 \times 8 = 0$ or, $R_{az} = -15 kN \downarrow$

$\sum F_z = 0$

or, $R_{az} + R_{bz} + R_{cz} = 0$

or, $R_{cz} = -R_{az} = 15 kN$

$\sum F_x = 0$ or, $R_{cx} = 15 kN$

Again, $\sum M_z = 0$ about line of action of R_{cz}

$$-R_{ay} \times 8 - 15 \times 5 = 0 \text{ or, } R_{ay} = \frac{-75}{8} = 9.37 \downarrow$$

$\sum F_y = 0$

or, $R_{ay} + R_{cy} = 0$

or, $R_{cy} = -R_{ay} = 9.37 \uparrow$

As all the reaction forces have been determined and now bar forces can be computed using method of joint. The forces in the three members *ad*, *dc* and *ac* need to be found out.

Observing the joint *d*, we see that there is no *y* component of force 15 kN and by symmetry,

$F_{ad} = -F_{dc}$

Again,

$-F_{adx} + 15 + F_{dcx} = 0$

$F_{dcx} = -\frac{15}{2} = -7.5$

$F_{adx} = 7.5$

Applying space truss theorem at joint *a* and *c*,
We get,

$F_{adz} = 15, \qquad F_{cdz} = -R_{cz} = -15$

At joint *a*,

$F_{adx} = -F_{acx} = -7.5, \; F_{acy} = 0 \; F_{acz} = 0$

Obviously, $F_{ady} = R_{ay} = 9.37$

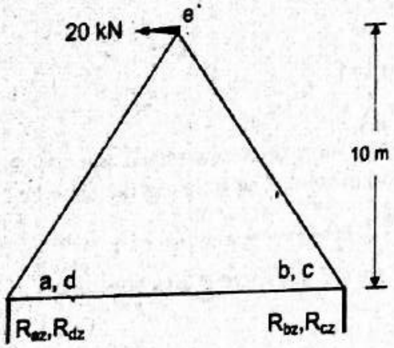
Similarly $F_{dcy} = -R_{cz} = -15$

Now the member forces are tabulated as below.

Member	Projection			Length	Component of Forces			Forces (kN)
	ℓ_x	ℓ_y	ℓ_z	m	F_x	F_y	F_z	
ad	4	5	8	10.25	7.5	9.37	15	19.21
cd	4	5	8	10.25	-7.5	-15	-15	-19.21
db	0	3	8	13.34	0	0	0	0
ab	4	8	0	8.94	0	0	0	0
bc	4	8	0	8.94	0	0	0	-7.5
ca	8	0	0	8	7.5	0	0	

EXERCISE 8.4

Ex. 8.1 Determine the reactions and bar forces for the space truss shown in Fig. 8.6



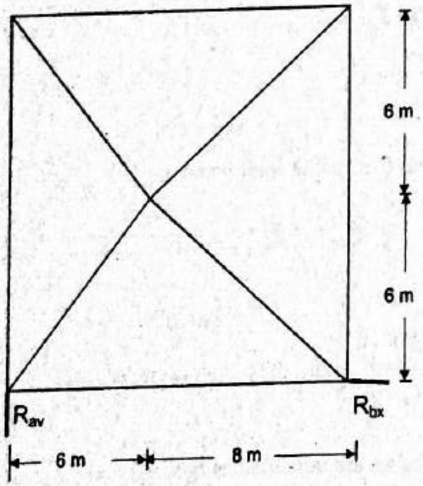


Fig. 8.6

Ans.

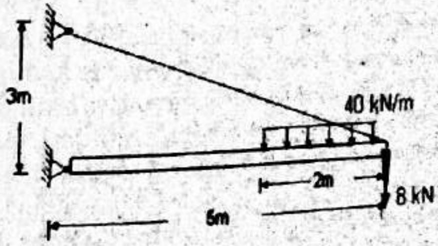
$F_{ab} = 4.28\ kN$	$F_{ac} = -9.36\ kN$
$F_{bc} = -4.28\ kN$	$F_{be} = 10.09\ kN$
$F_{cd} = -5.72\ kN$	$F_{ce} = 10.09\ kN$
$F_{da} = 4.28\ kN$	$F_{de} = -9.36\ kN$

9

QUESTIONS

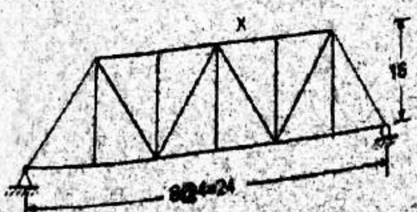
1. A steel section of cross section EI=3.36*10[11] Nmm is used as simply supported beam on an effective span of 4m. A weight of 250N falls from a height h to the midpoint of the beam. Calculate the value of height h in mm so that the maximum stress does not exceed 120 Mpa. Also calculate the maximum deflection induced to the beam by the fall.

2. Joints A,B and C are pin jointed. AB is a RCC beam of 250mm breadth 500mm depth. BC is of steel, 10mm in diameter. Find the vertical deflection at middle of the beam due to bending of the beam and axial force of steel member. Take Ec= 1*10[6] kg/cm and Es=2*10[6] kg/cm.

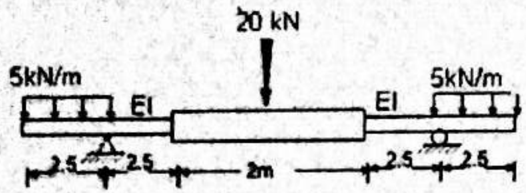


3. a) A simply supported beam of span 6m with an overhand of length 2m on both sides is loaded at 1m from the free ends with a concentrated force of magnitude 20KN each, besides udl of 5KN/m in middle third of the span. Calculate the deflection of the mid span using Macaulay' method.

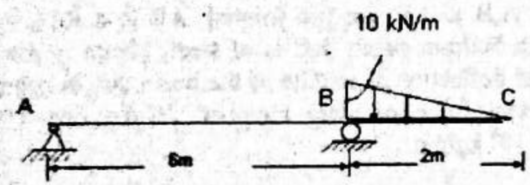
b) Draw influence line for axial force marked X of the truss shown.



4. a) Determine vertical deflection at middle of the beam.

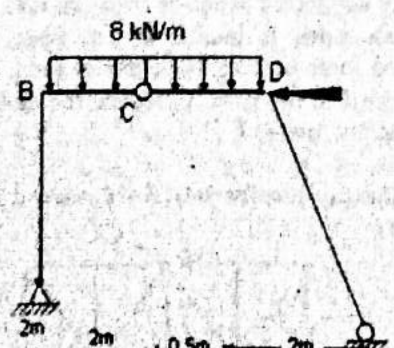


b) Using influence line diagram, calculate the bending moment at a section 1.5m from the left support.

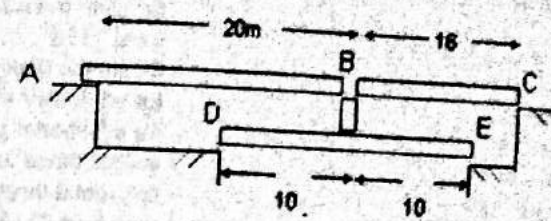


5. A three-hinged circular arch has a span of 80m and a rise of 10m. Two vertical loads 2t and 4t, with the smaller load leading, travels from left to right. Find the maximum positive bending moment at a section 15m from the right support. Also find the radial shear corresponding to this maximum bending moment.

6. a) Draw SF and BM diagrams for the frame loaded as shown.



b) Draw influence line for bending moment at a section 5m from support at D of the girder DE.



7. A symmetrical suspension bridge with a three hinged stiffening girder of span 120m and having a central dip of 12m is loaded with two point loads of magnitude 240kN and 300kN at a distance 25m and 80m respectively from the left end. Draw bending moment diagram for the girder and also find the location of the point of contra flexure in the diagram.